

Imperfect Competition, Long Lived Private Information, and the Implications for the Competition of High Frequency Trading

Su Li*

Department of Finance
Renmin University of China

First draft: November, 2011

Current draft: May 3, 2013

Abstract

I analyze the competition among strategic informed traders in an economy with a risky asset whose liquidation value is private information and follows a mean-reverting process. Instead of being one-shot, the private information is acquired by informed traders gradually. The unique linear equilibrium has an analytic form and is explicitly analyzed. In the limit of continuous trading, (i) the imperfectly competitive informed traders earn positive expected profits, contrary to the Bertrand-like results in Holden and Subrahmanyam (1992) and Foster and Visvanathan (1993), and (ii) they contribute significantly to price volatility and the fraction of total trading volume, whereas the monopolist in Chau and Vayanos (2008) has negligible contributions. These results can help (i) justify the co-existence of high frequency traders who employ very similar strategies, and (ii) provide new insights and policy suggestions regarding to the effect of high frequency competition.

*This paper is based on Chapter 1 of my dissertation at the Robert H. Smith School of Business, University of Maryland. I am especially grateful to my dissertation committee, Albert “Pete” Kyle (Chair), Steven L. Heston, Mark Loewenstein and Dilip Madan, for their guidance and constructive suggestions. I also thank Wei Li, Anna Obizhaeva, Bruno Biais, and Ioanid Rosu as well as participants at the seminars at the University of Maryland and Stony Brook University for their valuable discussions. All errors are my own.

1 Introduction

Will it be possible for traders with superior private information to earn strictly positive expected profits in a strong-form efficient market? Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) find that this is not possible and instead they reach a Bertrand-like result with zero expected profits although traders in their models compete through quantities (by submitting market orders). The intuition is that if there are at least two strategic traders receiving the same piece of private information, each trader tries to preempt the others given enough trading opportunities with the result that the private information is revealed instantaneously and each trader's expected profits quickly vanish to zero in the limit of continuous trading. Surprisingly, Chau and Vayanos (2008) find that positive expected profits are possible while the market is strong form efficient in a monopolistic setup. A strategic informed trader privately observes a flow of private information and chooses to trade aggressively on her information to push the price towards her valuation of the asset¹. In the limit, the information asymmetry disappears but the insider's profits converge to a positive constant. The difference between these models depends on the arrival of private information. Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) follow the assumptions in Kyle (1985) in which private information is one-shot and the risky asset has a fixed value and is to be liquidated at a predetermined date. In Chau and Vayanos (2008), the informed trader receives new information repeatedly, the fundamental value of the asset is stochastic, and trading takes place over an infinite horizon.

A question naturally arises, in an oligopolistic setting, if private information arrives

¹According to Chau and Vayanos (2008), the monopolist is "impatient" for three reasons: (1) time discounting, (2) public revelation of information, (3) mean-reversion of profitability.

gradually across the trading periods instead of being one-shot at the start of the trading sessions, will imperfect competition among informed traders still “kills” any trading profits in the limit of continuous trading? The answer to this question is critical to understand the evolution of financial markets. If the answer were “Yes” and conventional intuitions applied, as the computing technology push the speed of trading faster and faster, it will be even more difficult to justify the co-existence of several high frequency statistical arbitrageurs given that they are following similar strategies.

In this paper, I study the nature of imperfect competition among those traders. One of the purposes is to answer the question raised earlier regarding imperfect competition and the profitability of informed traders near continuous trading. The other purpose is to examine how competition affects other properties of the market such as price efficiency, the strategies of informed traders, market liquidity and trading volume, especially relative to the findings of the monopolist case in previous literature. In addition, the predictions from the model can be used to shed light on existing empirical facts, and provide new insights and policy suggestions with regards to high frequency trading.

In the model, there is a riskless bond and a risky asset. The liquidation value of the risky asset which follows a stochastic process can only be observed by informed traders. Like the “market order” model in Kyle (1985), multiple identically informed strategic traders and exogenous liquidity traders execute batched market orders against competitive risk neutral market makers. The informed traders receive new information each period and trading takes place until the asset is liquidated at a random date. I prove that there exists a unique linear equilibrium with a closed form solution, and derive analytical forms when trading becomes continuous. Not surprisingly, the combined trading of multiple informed traders is more

aggressive than the monopolistic trader in Chau and Vayanos, the equilibrium price is even more revealing of the informed trader's private information, and market depth improves as the number of informed traders increases. Oligopolistic imperfect competition makes the informed traders trade more aggressively than a monopolist, thus improving market efficiency and increasing aggregate trading volume. The effects of imperfect competition on market depth is slightly more difficult to interpret since it has two opposite effects. On the one hand, with increasing competition, initially the net order flow will contain more information relative to the noise trading, and therefore the adverse selection is more severe and market depth is worse. On the other hand, as market becomes more efficient, there is less private information conveyed by the demands of informed traders, and hence price is less sensitive to the net order flow. In the stationary state, I show that the second effect dominates and imperfect competition among informed traders help improving market liquidity.

Surprisingly, the model uncovers some important but unexpected results in the limit as the time interval between trades goes to zero. The first result concerns the profitability of the informed traders. If the intuition in Holden and Subrahmanyam (1992) is followed, one is inclined to draw a conclusion similar to theirs that at any point in time, any newly acquired private information is revealed into price instantaneously and profits goes to zero. However, such conventional wisdom does not apply in this model. As I prove formally, over one trading period, the aggregate profits of informed trading are of order Δt where Δt is the time interval between auctions. Therefore, the aggregate profits of the informed traders remain bounded away from zero. As the number of informed traders increases, their aggregate profits fall, tending to zero only as the number of informed traders becomes large. To be more specific, the aggregate profits near continuous trading is inversely proportional to the square root of

the number of informed traders. The result has the flavor of Cournot competition, not the flavor of Bertrand competition found in the one-shot private information model of Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993).

The second result concerns price efficiency. Although the model predicts that price is fully revealing near continuous trading, the level of efficiency differs from previous findings. If the variance of the private information not incorporated into price at each trading period is defined as the inverse measure of price efficiency, one will find that the variance goes to zero at a rate proportional to the time interval (Δt) between rounds of trading. This is much faster than the corresponding strong-form efficiency result in the Chau and Vayanos' model, where the convergence rate is proportional to the square root of the time interval ($\sqrt{\Delta t}$) between rounds of trading. The difference between the results of the two models hinges on how much more aggressively the oligopolistic informed traders exploit their informational advantage than the monopolistic informed trader. The informed traders in this model are very aggressive in exploiting pricing errors, with trading intensity over Δt to be proportional to 1, much higher than $\sqrt{\Delta t}$, the magnitude of trading intensity of the monopolist in Chau and Vayanos (2008) over the same trading period.

The third result concerns volume and volatility. The trading volume² of informed traders over Δt is of order $\sqrt{\Delta t}$, the same magnitude as the trading volume of liquidity traders. This implies that in the limit informed traders make a non-negligible contribution to total trading volume and price volatility, and the fraction of contribution converges to one as the number of informed traders becomes large. The result is novel since previous literature

²Trading volume is not well defined in continuous time Kyle's model since total variation of Brownian motion over any finite time is infinite.

(Kyle's model and many extensions) shows that the contributions to trading volume and price volatility by strategic informed traders (monopoly or oligopoly) are negligible compared to the contributions of liquidity traders.

These above theoretical results near continuous trading can generate some new empirical implications with regards to the competition of high frequency trading. In recent years, financial markets have witnessed rapid growth in high frequency trading³, made possible by the evolution of technology. High frequency traders are a subset of algorithmic traders. Those traders apply mathematical algorithms to either public or private statistical information, and they use fast computers to implement the algorithms, transmitting orders in a few milliseconds or less. High frequency traders contribute significantly to trading volume⁴. Despite aggressive competition with one another, high frequency traders remain profitable⁵. Kirilenko et al. (2011) document that high frequency traders are consistently profitable and they even turned a profit on the day on May 6, 2010 Flash Crash.

I focus on the type of high frequency traders who are pursuing low latency statistical arbitrage strategies⁶. According to Chlistalla (2011), these traders “seek to correlate between assets and try to profit from the imbalance in these correlations”. It might not be appropriate to label those traders as “informed” if one defines information as corporate news on “merger

³As pointed out by Duhigg in Stock Traders Find Speed Pays, in Milliseconds (*New York Times*, July 23, 2009), “Average daily volume has soared by 164 percent since 2005, according to data from NYSE. ..., stock exchanges say that a handful of high-frequency traders now account for more than half of all trades.”

⁴In “The Real Story of Trading Software Espionage” (*AdvancedTrading.com*, July 10, 2009), Iati mentions that “High-frequency trading firms, which represent approximately 2% of the trading firms operating in the U.S. markets today, account for 73% of all U.S. equity trading volume.”

⁵Iati's article states that “TABB group estimates that annual aggregate profits of low-latency arbitrage strategies exceed \$21 billion, spread out among the few hundred firms that deploy them.”

⁶There are other types of high frequency traders, such as the ones who are doing electronic market making, and ones who are front running other slow traders in the market. My main focus are the ones who are “informed” and consuming instead of providing liquidity, therefore I neglect other types of HFTs in the model.

and acquisition decisions” or “content of earnings announcements”. However, if those traders are faster and better than average market participants in gathering and processing market wide news including information on order flows and price movements on the security and any other correlated securities, to generate private signals which provide them with some sort of information advantage relative to the rest of market participants, it is reasonable to call them “informed”⁷. This model, together with Chau and Vayanos (2008), can generate very interesting and unique new insights with regards to the impact of high-frequency competition. With in a particular security where there is only one monopolistic HFT, the entry of another HFT with identical strategy will not only lead to the increase in price volatility, but also lead to a sharp increase in the fraction of trading volume participated by HFTs from a few percents in the monopoly to around 32% in the duopoly. Such insights cannot be generated by previous theoretical models on HFTs, whether static or dynamic. Empirical evidence on the effect of HFTs competition is rare, but a recent study by Breckenfelder (2013) uses ticker-level NASDAQ OMXS data. Because different HFTs can be identified in the data, Breckenfelder (2013) is able to exploit how the changes between monopolistic and duopolistic HFT within individual stocks affect the quality of the market. He finds that the intraday volatility increase by about 25% and a significant increase in market share of high-frequency traders, qualitatively confirming the predictions from this model.

After the flash crash of May 6, 2010, there was a policy proposal suggesting that batching orders less frequently can reduce the participation rate and profits of high frequency traders and improve market depth. My model implies that such a regulation could have the opposite

⁷Hendershott and Riordan (2011) find that the market orders by high frequency traders have information advantage.

effect and reduce liquidity instead. If there is a positive cost of being a high frequency trader, the number of high frequency traders might decrease with less frequent order batching, with the result that less competition will lead to market being less liquid.

This paper belongs to the literature on strategic trading with asymmetric information. In the pioneering work of Kyle (1985), a monopolistic insider uses liquidity traders as camouflage, reveals her private information gradually, and exploits her monopoly power over time when facing a competitive risk neutral market maker. In the subsequent extensions by Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993), due to the imperfect competition among identically informed traders, almost all private information is revealed only after a few trading rounds. Foster and Viswanathan (1994) replace homogenous private information with a hierarchical information structure to study the learning heterogeneously informed traders. Foster and Viswanathan (1996) relax the assumption even further to allow for a more general correlation structure among the signals received by multiple informed traders. They show the initial correlation among the signals has a strong effect on the trading strategies and informativeness of prices. Traders initially compete aggressively on the common part of the private information and later play a “waiting” game by making smaller bets and trying to infer private information exclusive to others. Back et al. (2000) solve a similar problem in continuous time and derive a closed-form solution.

The traders in my model exploit their private signals via market orders. Rosu (2009) directly models the limit order book. He also finds a similar prediction that higher competition causes smaller price impact. However, competition in his model is measured by how fast traders arrive in the market whereas in my model, competition is measured by the number of identically informed traders. In his model, traders trade for liquidity reasons and

there is no asymmetric information, whereas the motive for trade in my model comes from informational advantage.

Martinez and Rosu (2011) study a very similar problem. They tackle the problem directly in setup similar to that of Back (1992) and focus on non-stationary equilibrium. In order to generate linear equilibrium in continuous time, they assume an informed trader to have uncertainty aversion regarding the level of the asset value (Informed traders care more about the change in the value of asset than the level) and impose a technological constraint on the market maker. My paper, in contrast, does not require any extra assumptions and hence is more general.

In my model, high frequency traders are risk neutral. Therefore, the model cannot explain the phenomena that high frequency traders reverse their inventories frequently, and move in and out of short-term positions very quickly. Future work may explain pattern of mean-reverting inventories by making the traders risk averse instead of risk neutral.

The paper is organized as follows. In Section II, I describe the model, solve the linear equilibrium, and prove its uniqueness. Section III characterizes the equilibrium near continuous trading. Section IV shows some comparative statics results. Section V concludes.

2 The Model

Assumption 1: Securities

I consider an economy with a single consumption good. There are a riskless bond with zero interest rate and a non-dividend paying risky asset with a liquidation value v_n which evolves stochastically. Trading takes place from $t = 0$ to $t = +\infty$ at the discrete points $t_n = n\Delta t$ ($n = 0, 1, 2, \dots$), until the risky asset is liquidated where Δt is the time interval

between the auctions. At the end of each period, there is a probability $p = 1 - \exp(-r\Delta t)$ that the risky asset is liquidated. I further assume the riskless bond is in perfectly elastic supply. The liquidation value v_n follows a mean-reverting process or random walk:

$$v_n - v_{n-1} = \kappa(\bar{v} - v_{n-1})\Delta t + \epsilon_{v,n}. \quad (1)$$

In the above specification, κ determines the adjustment speed of the liquidation value v_n to its long run fixed target \bar{v} . κ is assumed to be greater than or equal to zero such that the prices are stationary. If $\kappa = 0$, then v_n follows a random walk. The innovation $\epsilon_{v,n}$ is independently and normally distributed with mean zero and variance $\sigma_v^2\Delta t$.

Assumption 2: Market Participants and Information Structure

The risk neutral market participants consist of a competitive market maker, M (M is a positive integer) informed strategic traders, and a number of liquidity traders. The informed traders are each assumed to be able to perfectly observe the liquidation value v_n at each period.

At each period, both the informed traders and liquidity traders submit market orders to the market maker. The liquidity traders' order is denoted by u_n , which is normally distributed with mean zero and variance $\sigma_u^2\Delta t$. I further assume u_n is uncorrelated with $\epsilon_{v,n}$. I denote the market order submitted by the j^{th} informed trader at the n^{th} period ($t = n\Delta t$) by $x_{j,n}$. In equilibrium, informed traders' demands should be identical at each period ($x_{1,n} = \dots = x_{M,n} = x_n$) because of symmetry argument.

Assumption 3: Timing of events

I assume at the n^{th} period, the informed traders and the noise traders submit their

demands before new private information arrives. After submitting their market orders, the informed traders observe ϵ_n and thus v_n . The market maker observes the total order imbalance $y_n = \sum_{j=1}^M x_{j,n} + u_n$, then sets the price p_n equal to the expected value of the asset based on the history of order flows, and takes the other side of the trade. At the end of the period, there is a probability $p = 1 - \exp(-r\Delta t)$ that the liquidation value v_n is public announced, the risky asset is liquidated and investors profits are realized. Conditional on that the asset has not yet been liquidated at the end of the n^{th} period, $I_n^i \equiv \{y_\tau, v_\tau | 0 \leq \tau \leq n\}$ is each informed trader's information set, and $I_n^m = \{y_\tau | 0 \leq \tau \leq n\}$ is the market makers' information set.

Pricing

Since the market maker is assumed to be competitive and risk neutral, therefore, at period n she sets the price p_n equal to the expected value of the asset after she receives the total batched market order $y_n = x_{1,n} + \dots + x_{M,n} + u_n$. Therefore,

$$p_n = \mathbb{E}\left[\sum_{n'=n}^{+\infty} (1 - \exp(-r\Delta t)) \exp(-r(n' - n)\Delta t) v_{n'} | I_{n-1}^m, y_n\right], \quad (2)$$

where $(1 - \exp(-r\Delta t)) \exp(-r(n' - n)\Delta t)$ is the probability that the asset is liquidated at the end of the n' th period.

Lemma 2.1: *The price p_n is a linear function of the market maker's expectation of the current liquidation value of the risky asset $\mathbb{E}[v_n | I_n^m]$:*

$$p_n = \frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} \mathbb{E}[v_n | I_n^m] + \frac{\kappa\Delta t \bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)}. \quad (3)$$

Proof: See Appendix A.

Optimization

Suppose the risky asset were liquidated at a random future date $\nu\Delta t$. Given that the asset has not been liquidated at $n\Delta t \leq \nu\Delta t$, the j^{th} informed trader's profits that accrue to her from period n should equal to the difference between the value of her position ($\sum_{n \leq \tau \leq \nu} v_\nu x_{j,\tau}$) and the cost of this position ($\sum_{n \leq \tau \leq \nu} p_\tau x_{j,\tau}$):

$$\pi_{j,n} = \sum_{n \leq \tau \leq \nu} (v_\nu - p_\tau) x_{j,\tau}. \quad (4)$$

Since informed traders are risk neutral, at the n^{th} period, the j^{th} informed trader tries to maximize her expected trading profits:

$$\begin{aligned} \mathbb{E}[\pi_{j,n} | I_{n-1}^j] &= E\left[\sum_{n \leq \tau \leq \nu} (v_\nu - p_\tau) x_{j,\tau} | I_{n-1}^j\right] \\ &= \mathbb{E}\left[\sum_{n'=n}^{+\infty} (1 - \exp(-r\Delta t)) \exp(-r(n' - n)\Delta t) \left(\sum_{\tau=n}^{n'} x_{j,\tau} (v_{n'} - p_\tau)\right) | I_{n-1}^j\right] \end{aligned} \quad (5)$$

Lemma 2.2: *The j^{th} informed trader's objective function can be written as:*

$$\max_{x_{j,n' \geq n}} \mathbb{E}\left[\sum_{n'=n}^{+\infty} \exp(-r(n' - n)\Delta t) x_{j,n'} \left(\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} v_{n'} + \frac{\kappa\Delta t \bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} - p_{n'}\right) | I_{n-1}^j\right]. \quad (6)$$

Proof: See Appendix A.

2.1 Equilibrium Concept

The equilibrium concept in this paper is similar to one in the previous literature. I follow Foster and Viswanathan (1996) closely here and let $X_j = (x_{j,1}, \dots, x_{j,\nu})$ (for each j) and $P = (p_1, \dots, p_\nu)$ represent the strategy functions were the asset liquidated at $\nu\Delta t$. A Bayesian Nash equilibrium of the trading game is a $M + 1$ vector of strategies (X_1, \dots, X_M, P) such that:

1. For any $j = 1, \dots, M$ and all $n = 1, \dots, \nu$, I have for $X'_j = (x'_{j,1}, \dots, x'_{j,\nu})$

$$\mathbb{E}[\pi_{j,n}(X_1, \dots, X_j, \dots, X_M, P)|I_{n-1}^i] \geq \mathbb{E}[\pi_{j,n}(X_1, \dots, X'_j, \dots, X_M, P)|I_{n-1}^i] \quad (7)$$

2. For all $n = 1, \dots, \nu$, I have

$$p_n = \frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} \mathbb{E}[v_n|I_n^m] + \frac{\kappa\Delta t\bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)}. \quad (8)$$

Therefore, the market maker sets the price equal to the expected value of the risky asset conditional on her information set inferred from the order flow. Each risk neutral informed trader, taking as given the price process set by the market maker and the strategies of other informed traders, submits market orders to maximize the expected profits taking into account the effect on the price.

I restrict attention to stationary linear Markov equilibrium. In order to set the price p_n which takes a linear form in equation (3), the market maker has to solve the inference problem about v_n . I then conjecture that informed trader j 's optimal strategy at period n is to submit demands which depend linearly on the pricing error defined as the difference between v_{n-1} and the market maker's conditional estimation on v_{n-1}

$$x_{j,n} = \beta_j(v_{n-1} - \hat{v}_{n-1}), \quad (9)$$

where $\hat{v}_{n-1} = \mathbb{E}[v_{n-1}|I_{n-1}^m]$ to maximize her expected profits.

2.2 The Market Maker's Inference Problem

To solve the market maker's inference problem, I use Kalman filtering. Conjecture that at the end of the $(n-1)^{th}$ period, the market maker believe v_{n-1} to be normally distributed

with mean \hat{v}_{n-1} and variance Σ_v ⁸. Then, at the n^{th} period, after observing the net order imbalance y_n , the market maker updates her belief about v_{n-1} in the form of

$$\begin{aligned} v_{n-1} &= \mathbb{E}[v_{n-1}|I_{n-1}] + \frac{\lambda}{1 - \kappa\Delta t} y_n + \eta_n \\ &= \hat{v}_{n-1} + \frac{\lambda}{1 - \kappa\Delta t} (X_n + u_n) + \eta_n, \end{aligned} \quad (10)$$

where the constant λ is the inference parameter for the market maker to be derived next. Since v_n follows a mean-reverting process (random walk if $\kappa = 0$) described in Equation (1), v_n has the following expression:

$$v_n = (1 - \kappa\Delta t)\hat{v}_{n-1} + \kappa\Delta t\bar{v} + \lambda y_n + (1 - \kappa\Delta t)\eta_n + \epsilon_{v,n}. \quad (11)$$

Therefore, the market maker's posterior belief about v_n is normally distributed with mean

$$\mathbb{E}[v_n|I_n^m] = \hat{v}_n = (1 - \kappa\Delta t)\hat{v}_{n-1} + \kappa\Delta t\bar{v} + \lambda y_n \quad (12)$$

and variance

$$\text{Var}(v_n|I_n^m) = \text{Var}((1 - \kappa\Delta t)\eta_n + \epsilon_{v,n}) = (1 - \kappa\Delta t)^2 \text{Var}(\eta_n) + \sigma_v^2 \Delta t. \quad (13)$$

Stationary condition requires that $\text{Var}(v_n|I_n^m) = \Sigma_v$.

Lemma 2.3: *Given the trading strategy of the informed traders defined in equation (9), the market maker's inference parameter λ is given by*

$$\lambda = \frac{(1 - \kappa\Delta t)\Sigma_v \sum_{j=1}^M \beta_j}{(\sum_{j=1}^M \beta_j)^2 \Sigma_v + \sigma_u^2 \Delta t}, \quad (14)$$

and the variance of the market maker's belief on v_n satisfies the equation

$$\frac{(1 - \kappa\Delta t)^2 \Sigma_v \sigma_u^2 \Delta t}{(\sum_{j=1}^M \beta_j)^2 \Sigma_v + \sigma_u^2 \Delta t} + \sigma_v^2 \Delta t = \Sigma_v. \quad (15)$$

⁸ Σ_v is strictly greater than $\sigma_v^2 h$ since the informed traders observe $\epsilon_{v,n-1}$ after they submit the market order at the $(n-1)$ th period.

Proof: See Appendix B.

2.3 The Informed Traders' Optimization Problem

At the beginning of each period, each informed trader submits market orders given the price process generated by the market maker to maximize the present value of the expected profits scaled by $\frac{1-\exp(-r\Delta t)}{1-\exp(-r\Delta t)(1-\kappa\Delta t)}$. The j^{th} informed trader's optimization problem becomes

$$\mathbb{E}\left[\sum_{n'=n}^{+\infty} x_{j,n'}(v_{n'} - \hat{v}_{n'})e^{-r(n'-n)\Delta t} | I_{n-1}^i\right]. \quad (16)$$

In the optimization problem, the informed trader takes into account of how his trading and the trading by other informed traders affect the market price. I conjecture that the value function for the j^{th} informed trader is quadratic with respect to $v_{n-1} - \hat{v}_{n-1}$ and takes the form of

$$V_j(v_{n-1} - \hat{v}_{n-1}) = B_j(v_{n-1} - \hat{v}_{n-1})^2 + C_j. \quad (17)$$

Later, I will prove the quadratic form is sustained and valid in Lemma 2.4. The value function must satisfy the following Bellman equation:

$$V_j(v_{n-1}, \hat{v}_{n-1}) = \max_{x_{j,n}} \mathbb{E}[x_{j,n}(v_n - \hat{v}_n) + e^{-r\Delta t} V_j(v_n, \hat{v}_n) | I_{n-1}^j]. \quad (18)$$

The solution of the above Bellman equation is provided in the following theorem:

Lemma 2.4: *Given the price process set by the market maker, each informed trader's strategy (equation (9)) is characterized by the same trading intensity parameter⁹ β , given by*

$$\beta = \frac{(1 - 2e^{-r\Delta t} B\lambda)(1 - \kappa\Delta t)}{\lambda(M + 1 - 2Me^{-r\Delta t} B\lambda)}. \quad (19)$$

⁹Since at equilibrium, informed traders choose identical strategy. From this point, I suppress the subscript j .

Equation (18) has a quadratic solution of the form $V(v_{n-1}, \hat{v}_{n-1}) = B(v_{n-1} - \hat{v}_{n-1})^2 + C$ where B and C satisfy the following set of equations:

$$B = \frac{(1 - \kappa\Delta t)^2(1 - e^{-r\Delta t}B\lambda)}{\lambda(1 + M - 2Me^{-r\Delta t}B\lambda)^2} \quad (20)$$

and

$$C = \frac{e^{-r\Delta t}B(\lambda^2\sigma_u^2 + \sigma_v^2)\Delta t}{1 - e^{-r\Delta t}}. \quad (21)$$

Proof: See Appendix B.

2.4 Equilibrium

Proposition 2.1: *There exists a unique linear Markovian equilibrium characterized by five parameters λ , Σ_v , β , B and C which satisfy the system of five nonlinear equations: (14), (15), (19), (20) and (21). The expressions for Σ_v , β , λ , B and C are given by:*

$$\Sigma_v = \frac{M(1 - 2q_1) + 1}{M(1 - 2q_1) + 1 - (1 - \kappa\Delta t)^2} \sigma_v^2 \Delta t, \quad (22)$$

$$\beta = \frac{\sigma_u}{\sigma_v} \sqrt{\frac{(1 - 2q_1)(M(1 - 2q_1) + 1 - (1 - \kappa\Delta t)^2)}{M(M(1 - 2q_1) + 1)}} \quad (23)$$

$$\lambda = \frac{(1 - \kappa\Delta t)\sigma_v\sqrt{M(1 - 2q_1)}}{\sigma_u\sqrt{M(1 - 2q_1) + 1}\sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}}, \quad (24)$$

$$B = \frac{e^{r\Delta t}q_1\sigma_u\sqrt{M(1 - 2q_1) + 1}\sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}}{(1 - \kappa\Delta t)\sigma_v\sqrt{M(1 - 2q_1)}} \quad (25)$$

and

$$C = \frac{q_1\sigma_u\sigma_v\Delta t}{1 - e^{-r\Delta t}} \left[\frac{(1 - \kappa\Delta t)\sqrt{M(1 - 2q_1)}}{\sqrt{M(1 - 2q_1) + 1}\sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}} + \frac{\sqrt{M(1 - 2q_1) + 1}\sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}}{(1 - \kappa\Delta t)\sqrt{M(1 - 2q_1)}} \right] \quad (26)$$

where

$$\begin{aligned}
q_1 &= \frac{M+1}{3M} & (27) \\
&- \frac{1}{6M} \sqrt[3]{(M+1)^3 + (9-18M)Z + \sqrt{((M+1)^3 + (9-18M)Z)^2 - ((M+1)^2 - 3Z)^3}} \\
&- \frac{1}{6M} \sqrt[3]{(M+1)^3 + (9-18M)Z + \sqrt{((M+1)^3 + (9-18M)Z)^2 + ((M+1)^2 - 3Z)^3}},
\end{aligned}$$

$Z = e^{-r\Delta t}(1 - \kappa\Delta t)^2 < 1$ and $M \geq 2$.

Proof: See Appendix B.

3 Asymptotic Properties of Equilibrium in the Limit of Continuous Trading

3.1 Research Questions

It is important to study how imperfect competition affects the properties of equilibrium, especially when the frequency of trading becomes very large approaching to the limit of continuous trading. Firstly, it is intuitively to believe that increasing competition will make the already “impatient” monopolistic trader in Chau and Vayanos (2008) even more “impatient”. The aggregate trading of multiple informed traders competing strategically should be more aggressive than in the monopolist case and therefore price should become more efficient. Chau and Vayanos find that when the time interval between rounds of trading is small, the trading volume of the informed monopolist over Δt is of order $\Delta t^{3/4}$, and the variance of private information not incorporated into price Σ_v is proportional to $\sqrt{\Delta t}$. It is interesting to ask how does the more aggressive trading by informed traders affect the trading volume and price efficiency? Will the oligopolistic informed traders contribute the same magnitude of trading volume as in the monopolistic case or some magnitude higher?

If higher, how will that affect the level of price efficiency?

Secondly, does more competition improve market depth or not? As shown in the non-stationary setup in Holden and Subrahmanyam (1992), imperfect competition has two opposite effects. The market is less liquid in the beginning of the trading sessions since the net order flow contains more private information relative to the noise, and the market maker has to set price more sensitive to the order flow. After most information is revealed in the remaining of trading sessions, the market becomes very deep since price is already very efficient and there is less information asymmetry between the informed traders and market maker. Since I am focusing on the stationary state equilibrium, it is reasonable to believe that the second effect dominates and market depth should be improved.

Thirdly, Foster and Viswanathan (1993) show that identically informed traders' profits converge to zero in a continuous trading limit since by trading more frequently the traders have more opportunities to preempt other traders. In their model and the model of Holden and Subrahmanyam (1992), private information is one-shot. It is important to examine whether this Bertrand-like result still holds in the model where private information is observed gradually. Since the informed traders' profitability hinges on the liquidity trader's ability to destabilize the price, the informed traders earn positive expected profits as long as the price impact remains non-zero. The market maker cannot set price impact to zero because there is always new information coming in and the market maker has to be cautious on the information embedded in the net order flow. As long as the price impact is strictly positive in the limit, informed traders' profits should remain positive and bounded away from zero.

To answer the above questions, I derive the asymptotic properties of the equilibrium near

continuous trading in the following theorem.

3.2 Asymptotic Properties of Equilibrium

Proposition 2.2: In the limit of continuous trading ($\Delta t \rightarrow 0$), the asymptotic behaviors of Σ_v , β , λ , B and C are given by:

$$\lim_{\Delta t \rightarrow 0} \frac{\Sigma_v}{\Delta t} = \frac{\sigma_v^2(1 + M(1 - 2q_0))}{M(1 - 2q_0)} \quad (28)$$

$$\lim_{\Delta t \rightarrow 0} \beta = \frac{\sigma_u(1 - 2q_0)}{\sigma_v \sqrt{1 + M(1 - 2q_0)}} \quad (29)$$

$$\lim_{\Delta t \rightarrow 0} \lambda = \frac{\sigma_v}{\sigma_u} \sqrt{\frac{1}{1 + M(1 - 2q_0)}} \quad (30)$$

$$\lim_{\Delta t \rightarrow 0} B = \frac{q_0 \sigma_u}{\sigma_v} \sqrt{1 + M(1 - 2q_0)} \quad (31)$$

$$\lim_{\Delta t \rightarrow 0} C = \frac{q_0 \sigma_u \sigma_v}{r} (\sqrt{1 + M(1 - 2q_0)} + \sqrt{\frac{1}{1 + M(1 - 2q_0)}}). \quad (32)$$

where

$$\begin{aligned} q_0 &= \frac{M + 1}{3M} \\ &- \frac{1}{6M} \sqrt[3]{(M + 1)^3 - 18M + 9 + \sqrt{((M + 1)^3 - 18M + 9)^2 - (M^2 + 2M - 2)^3}} \\ &- \frac{1}{6M} \sqrt[3]{(M + 1)^3 - 18M + 9 - \sqrt{((M + 1)^3 - 18M + 9)^2 - (M^2 + 2M - 2)^3}}. \end{aligned} \quad (33)$$

Proof: See Appendix B.

From the above theorem, the parameter which measures the uncertainty of the market maker about the liquidation value of the risky asset, Σ_v , converges to 0 when Δt goes to 0. The value Σ_v is also the variance of private information not incorporated into price at each period. The notation $\Sigma_v \sim \Delta t$ means that when trading is continuous, all private information is reflected in the price, information asymmetry disappears, and the market is

strong form efficient. Although we reach the same conclusion regarding market efficiency as in Chau and Vayanos' model, there is still some difference in the level of price efficiency for small Δt . Since $\Sigma_v \sim \sqrt{\Delta t}$ for the case of a monopolistic informed trader and $\Sigma_v \sim \Delta t$ for the imperfectly competitive case, we have $\frac{\Sigma_v(M=1)}{\Sigma_v(M>1)} \sim \frac{1}{\sqrt{\Delta t}}$ with the ratio converging to infinity when $\Delta t \rightarrow 0$. Therefore, the equilibrium price with $M \geq 2$ informed traders is even more revealing of the informed traders' private information than the monopolist case. We should also expect that trading intensity is qualitatively different. In Chau and Vayanos, when there is only one monopolistic informed trader, $\beta \sim \sqrt{\Delta t}$. However, β is of order 1 when the market is populated with multiple informed traders. This implies that imperfect competition makes traders to exploit the pricing errors much more aggressively and in turn brings information into the price much more quickly. Therefore, market makers learn more from the order flows and set more efficient price.

Next, I examine the trading volume contributed by informed traders to check whether it is comparable to the trading volume of liquidity traders in continuous trading. One can tell that β is of order 1 and $|v_{n-1} - \hat{v}_{n-1}| \sim \sqrt{\Delta t}$. Then over one trading period, the absolute aggregate trading volume of an informed trader $|x_n| = \beta|v_{n-1} - \hat{v}_{n-1}|$ is of order $\sqrt{\Delta t}$. Since the trading volume contributed by liquidity trader is of the same order $|u_n| = \sigma_u \sqrt{\Delta t}$, it follows that the informed traders generate a non-negligible fraction of total trading volume because the ratio $\frac{|x_n|}{|u_n|}$ converges to a positive constant bounded away from zero. The next theorem derives the fraction of trading volume contributed by informed traders.

Proposition 2.3: *Define $\xi_M = \frac{M|x_n|}{M|x_n|+|u_n|+|y_n|}$ to be the fraction of trading volume contributed by the informed traders. The value of ξ_M can be expressed as*

$$\xi_M = \frac{M\sqrt{1-2q_1}}{M\sqrt{1-2q_1} + 1 + \sqrt{M^2(1-2q_1) + 1}}. \quad (34)$$

In the limit of continuous trading $\Delta t \rightarrow 0$, we have

$$\xi_M^0 = \lim_{\Delta t \rightarrow 0} \xi_M = \frac{M\sqrt{1-2q_0}}{M\sqrt{1-2q_0} + 1 + \sqrt{M^2(1-2q_0) + 1}} \quad (35)$$

which depends only on the number of informed traders and is strictly greater than zero.

Proof: See Appendix B.

In most dynamic market microstructure models, the informed traders' trade does not have a diffusion component which contributes to volatility in the limit of continuous trading since their fraction of trading volume is zero. In my model, as illustrated above, informed traders contribute significantly to trading volume, and therefore they should contribute significantly to price volatility as well. It is trivial to write Δp_n as

$$\begin{aligned} \Delta p_n &= p_n - p_{n-1} = \frac{(1 - \exp(-r\Delta t))}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} \Delta \hat{v}_n \\ &= \frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} \lambda(M\beta(v_{n-1} - \hat{v}_{n-1}) + u_n). \end{aligned} \quad (36)$$

The price variance in the stationary state can therefore be written as

$$\frac{\text{Var}(\Delta p_n)}{\Delta t} = \left(\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} \right)^2 \frac{\lambda^2(M^2\beta^2\Sigma_v + \sigma_u^2 h)}{\Delta t}. \quad (37)$$

The next theorem illustrates the contribution of price variance by informed traders and liquidity traders.

Proposition 2.4: *In the continuous trading limit ($\Delta t \rightarrow 0$), the price variance $\lim_{\Delta t \rightarrow 0} \frac{\text{Var}(\Delta p_n)}{\Delta t}$ can be decomposed into two components: (i) a contribution from informed traders given by $(\frac{r}{r+\kappa})^2 \beta^2 \lambda^2 \frac{\sigma_v^2(1+M(1-2q_0))}{M(1-2q_0)}$; (ii) a contribution from liquidity traders given by $(\frac{r}{r+\kappa})^2 \frac{\sigma_v^2}{1+M(1-2q_0)}$. The total price variance which is the sum of these two components, is $(\frac{r}{r+\kappa})^2 \sigma_v^2$, independent of the number of traders.*

Proof: (i) and (ii) are trivial to prove. To prove the last point regarding total price variance, observe that

$$\begin{aligned}
\lambda^2(M^2\beta^2\Sigma_v + \sigma_u^2) &= \frac{\sigma_v^2}{\sigma_u^2} \frac{1}{1 + M(1 - 2q_0)} \left[\frac{M^2(1 - 2q_0)^2}{1 + M(1 - 2q_0)} \frac{\sigma_u^2}{\sigma_v^2} \frac{\sigma_v^2(1 + M(1 - 2q_0))}{M(1 - 2q_0)} \right. \\
&\quad \left. + \sigma_u^2 \right] \\
&= \frac{\sigma_v^2}{\sigma_u^2} \frac{1}{1 + M(1 - 2q_0)} [M(1 - 2q_0)\sigma_u^2 + \sigma_u^2] \\
&= \sigma_v^2.
\end{aligned} \tag{38}$$

Finally, I examine each informed trader's profitability. The expected profits can be written as

$$\mathbb{E}\left[\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} V(v_{n-1}, \hat{v}_{n-1})\right] = \frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} (B \mathbb{E}[(v_{n-1} - \hat{v}_{n-1})^2] + C). \tag{39}$$

The term $B \mathbb{E}[(v_{n-1} - \hat{v}_{n-1})^2] = B\Sigma_v$ converges to 0 when $\Delta t \rightarrow 0$. But $\frac{(1 - \exp(-r\Delta t))}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} C$ converges to a positive constant from Proposition 2.2. Hence, competition makes the aggregate profits fall, but it does not drive profits to zero. The results are in sharp contrast with the ones found in Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) although sharing a similar result in terms of market efficiency.

If we further assume that each informed trader has to pay a fixed cost c to acquire the stream of private signals, then whether each informed trader's decision to acquire the signals and trade hinges on whether her expected utility (expected profits) from trading exceeds her utility from not trading. Since the traders are risk neutral, the expected utility from trading is $\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} (B\Sigma_v + C)$ and the utility from not trading is c . Therefore, the equilibrium number of informed traders is the largest integer that satisfies the following

condition:

$$\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)}(B\Sigma_v + C) \geq c. \quad (40)$$

In the limit of continuous trading, the inequality condition becomes

$$\frac{q_0\sigma_u\sigma_v}{r+k}(\sqrt{1 + M(1 - 2q_0)} + \sqrt{\frac{1}{M(1 - 2q_0)}}) \geq c. \quad (41)$$

3.3 Properties of the Perfectly Competitive Equilibrium

I examine another class of asymptotic results by taking the limit as M goes to infinity. It is easy to verify that when $M \rightarrow +\infty$, $\lim_{M \rightarrow \infty} M^2 q_0 = 1$. Then substituting $\frac{1}{M^2}$ for q_0 in Proposition 2.2, I can derive the properties of the perfectly competitive equilibrium in the limit of continuous trading in the next theorem.

Proposition 2.5: In the perfectly competitive case (i.e., when the number of traders goes to infinity), the asymptotic properties of the equilibrium now becomes:

$$\lim_{\Delta t \rightarrow 0, M \rightarrow +\infty} \frac{\Sigma_v}{\Delta t} = \sigma_v^2 \quad (42)$$

$$\lim_{\Delta t \rightarrow 0, M \rightarrow +\infty} \sqrt{M}\beta = \frac{\sigma_u}{\sigma_v} \quad (43)$$

$$\lim_{\Delta t \rightarrow 0, M \rightarrow +\infty} \sqrt{M}\lambda = \frac{\sigma_v}{\sigma_u} \quad (44)$$

$$\lim_{\Delta t \rightarrow 0, M \rightarrow +\infty} M^{\frac{3}{2}}B = \frac{\sigma_u}{\sigma_v} \quad (45)$$

$$\lim_{\Delta t \rightarrow 0, M \rightarrow +\infty} M^{\frac{3}{2}}C = \frac{\sigma_u\sigma_v}{r} \quad (46)$$

$$\lim_{M \rightarrow +\infty} \xi_M = \frac{1}{2} \quad (47)$$

Since $\sigma_v^2\Delta t$ is the variance of new private information the informed traders learn at

each period, $\lim_{\Delta t \rightarrow 0, M \rightarrow +\infty} \frac{\Sigma_v}{\Delta t} = \sigma_v^2$ implies that in the perfectly competitive case, there is no information left on the table. The result on β suggests that although each individual trader's trading intensity can be infinitesimally small, the aggregate trading intensity can be very large as the number of traders increases. The results on λ and C suggest that as the number of informed traders increases, market depth improves, but aggregate expected profits fall, tending to zero only as the number of informed traders becomes large. The result on ξ_M , the fraction of trading volume from informed traders, can be arbitrarily close to $\frac{1}{2}$ as the number of traders is large enough. The fraction cannot go beyond $\frac{1}{2}$ since the total trading volume includes the contribution from the market maker.

4 Numerical Illustrations and Comparative Statics

In what follows, I numerically illustrate how information structure and imperfect competition among informed traders affect market efficiency, market liquidity, trading volume, price volatility, and expected profits of the informed traders. I also provide some empirical implications.

Since the parameter M is defined to be the number of informed traders, the model becomes a monopolist case if M is set to be 1 and an oligopolistic case when $M \geq 2$. Most of the comparative static analysis is concerned with the effect of changing M . One issue concerns the comparison between the duopolist case and monopolist case. The entry of an extra identical informed trader to the monopolist case will change the asymptotic properties of equilibrium in Chau and Vayanos' model dramatically. Another issue concerns within the oligopolistic situation. As we increase M from $M = 2$ to large values, the competition among informed traders increases. I examine how changing the intensity of competition affects the

properties of equilibrium.

Market Efficiency

I set the parameters such that $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$. The variable Σ_v measures the market maker's uncertainty about the liquidation value of the risk asset. It is therefore an inverse measure of price efficiency. $\Sigma_v = 0$ corresponds to the scenario where information asymmetry vanishes and the market is strong-form efficient. Since increasing the number of imperfect competitors makes traders willing to incorporate more private information into the price, the price should become more informative and hence a smaller Σ_v should be expected. As trading becomes more frequent (Δt is smaller), the noncooperative setting results in a more aggressive competition, making the already "impatient" informed traders even more "impatient". To illustrate these intuitions, I show how Σ_v varies with Δt and M in Figure 1(A).

As shown in the figure, Σ_v monotonically decreases with Δt for the monopolist case and for the oligopolist cases when $M = 2, 3$ and 10 . The value of Σ_v declines more rapidly for $M \geq 2$ than for $M = 1$. If we fix Δt and vary only the number of informed traders, Σ_v is found to be inversely related with the number of informed traders M .

Next I examine the asymptotic properties of Σ_v . According to Chau and Vayanos (2008), $\Sigma_v \sim \sqrt{\Delta t}$ as $\Delta t \rightarrow 0$ in the monopolist case $M = 1$. I prove in Proposition 2.2 that $\Sigma_v \sim \Delta t$ in the oligopolist case $M \geq 2$. In Figure 1(B), I show how the scaled value of Σ_v varies with Δt for different M . I scale Σ_v by $\sqrt{\Delta t}$ for $M = 1$ and Δt for $M \geq 1$. From the figure, $\frac{\Sigma_v}{\sqrt{\Delta t}}$ approaches to a constant for $M = 1$, confirming the asymptotic result obtained for the monopolistic trader. When $M \geq 2$, $\frac{\Sigma_v}{\Delta t}$ converges to a positive constant confirming the asymptotic property of Σ_v obtained in Proposition 2.2. Since the ratio $\frac{\Sigma_v(M=1)}{\Sigma_v(M \geq 2)} \frac{\sqrt{\Delta t}}{\Delta t} \rightarrow \infty$ as

Δt converges to zero, private information is revealed much more quickly and price becomes more efficient when there are multiple traders in the market.

Trading Intensity β

I have shown that, with increasing competition, more information is incorporated into price. Intuitively, one should expect that the aggregate trading intensity should be higher, and the fraction of trading volume contributed by informed traders should also be higher when M increases. I demonstrate numerically how each trader's trading intensity β and aggregate trading volume generated by the informed traders per period $\mathbb{E}[|X_n|]$ vary with Δt , respectively, for the cases when $M = 1, 2, 3$ and 10 in Figure 2.

As can be seen from the figure, when we compare monopoly $M = 1$ with duopoly $M = 2$, the increased competition between the two traders induces each duopolist to choose a higher trading intensity β as shown in Figure 2(A) when Δt is small. But the result reverses sign as we continue adding more informed traders when Δt is small. For $M \geq 2$, the more the informed traders, the less trading intensity from each individual trader. Each trader's optimal strategy is to exploit less the investment opportunity determined by the difference between the true valuation of the asset and the price set by the market maker. Consequently, each trader may actually trade less intensely as more traders become informed. In aggregate, competition does make the traders behave more aggressively since the aggregate trading intensity $M\beta$ increases as competition becomes more intensive. In other words, when Δt is small, for $M \geq 2$, the value of β is decreasing in M while the value of $M\beta$ is increasing in M . Figure 3 illustrates how the asymptotic trading intensity $\lim_{\Delta t \rightarrow 0} \beta$ and aggregate trading intensity $M \lim_{\Delta t \rightarrow 0} \beta$ vary with the number of informed traders. Although each individual trader trades less intensely, the aggregate trading intensity monotonically increases with M .

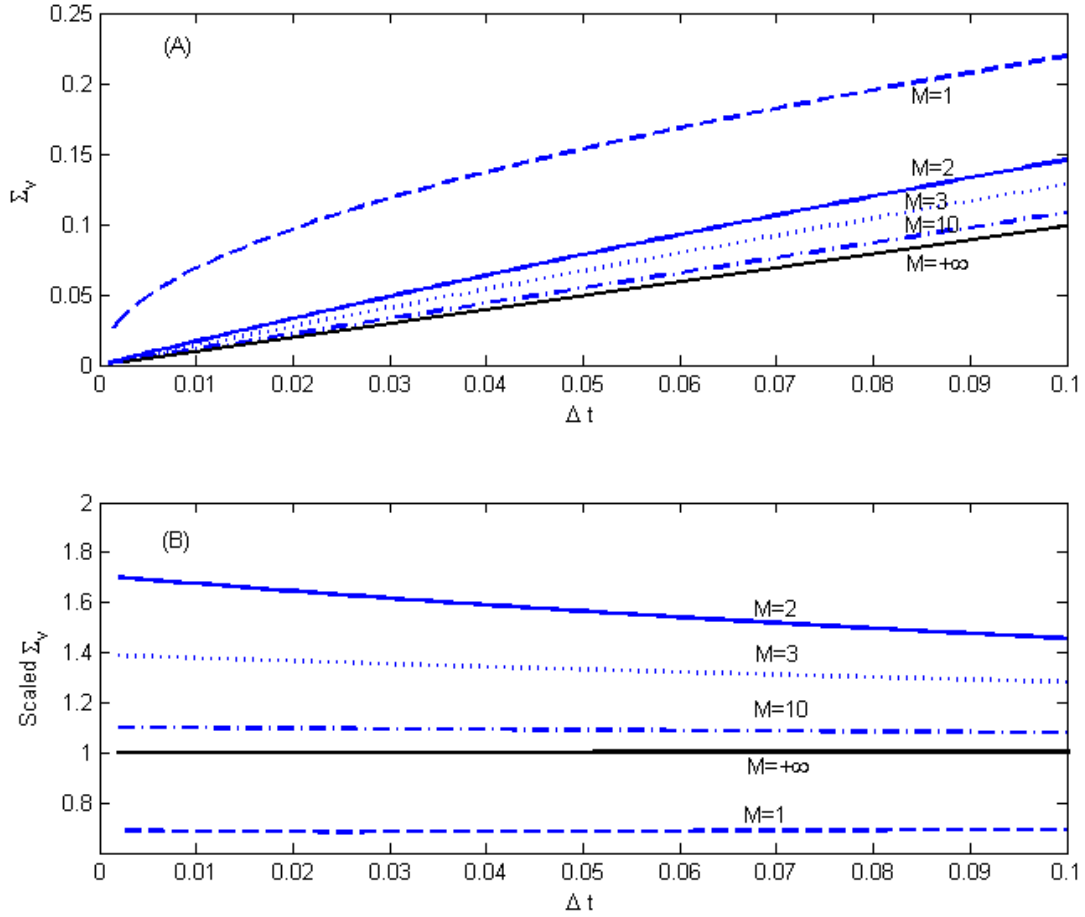


Figure 1: (A) Σ_v as a function of Δt for $M = 1$, $M = 2$, $M = 3$ and $M = 10$. (B) Scaled Σ_v as a function of Δt . Σ_v is scaled by $\sqrt{\Delta t}$ for $M = 1$, and by Δt for $M \geq 2$. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

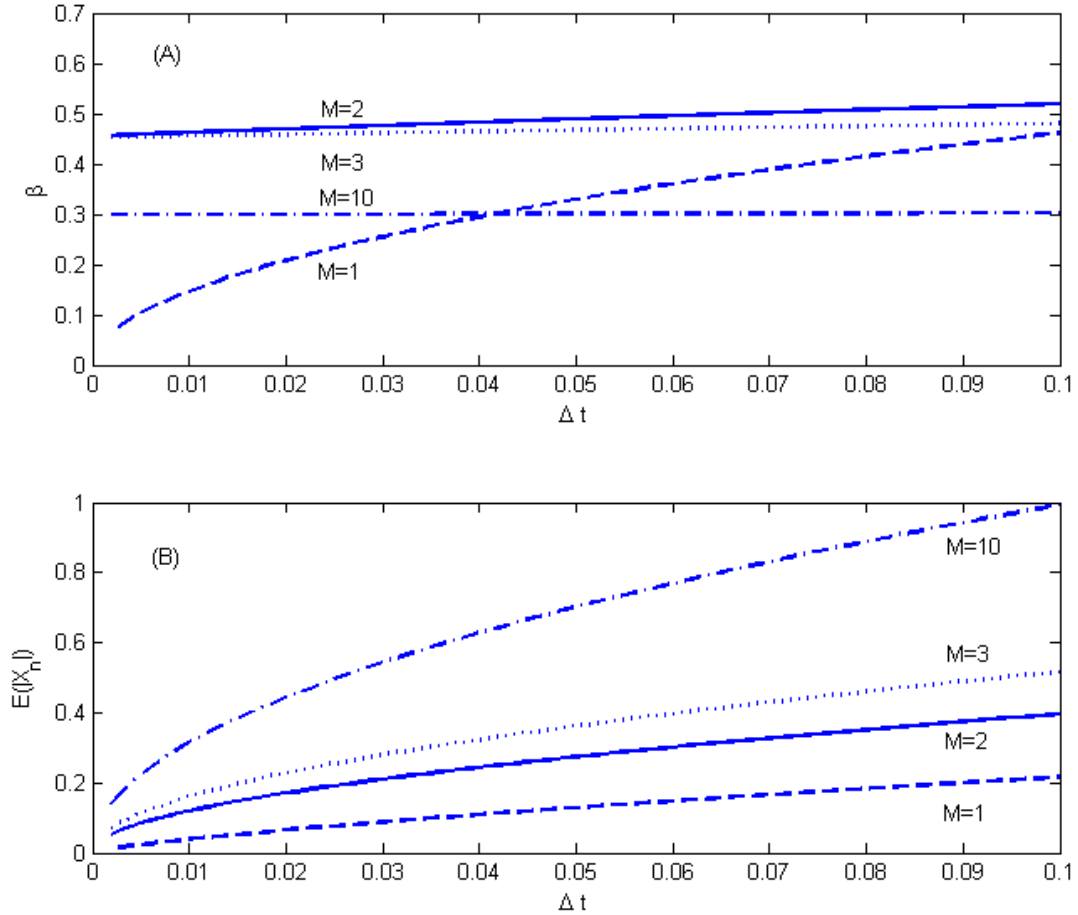


Figure 2: (A) The trading intensity parameter β as a function of Δt for $M = 1$, $M = 2$, $M = 3$ and $M = 10$. (B) The aggregate expected trading volume per period $\mathbb{E}[X_n]$ as a function of Δt for $M = 1$, $M = 2$, $M = 3$ and $M = 10$. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

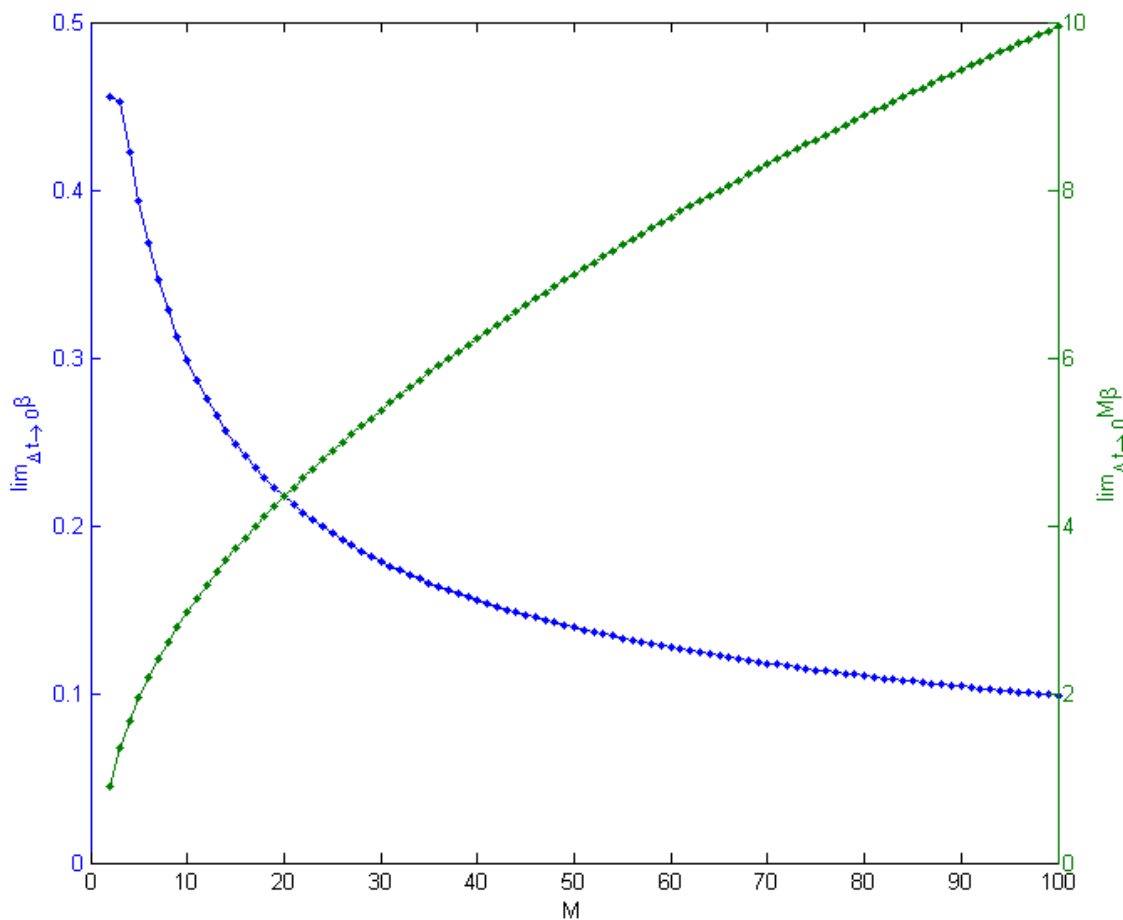


Figure 3: $\lim_{\Delta t \rightarrow 0} \beta$ and $\lim_{\Delta t \rightarrow 0} M\beta$ as functions of the number of informed traders.

Trading Volume

As shown in Figure 2(B), the expected aggregate quantity of informed trading per period $\mathbb{E}[|X_n|]$ monotonically increases with the number of traders M . Therefore, although each individual trader tends to submit lower demand when the market becomes more competitive, the aggregate trading volume which contains private information increases, conveying more information to the market maker.

The asymptotic properties of β and $\mathbb{E}[|X_n|]$ can be inferred from Figure 2. In the mo-

nopolist case, β is of order $\sqrt{\Delta t}$ and $\mathbb{E}[|X_n|] = \beta\sqrt{\Sigma_v}$ is of order $\Delta t^{\frac{3}{4}}$. In the imperfectly competitive case, β converge to a positive constant and $\mathbb{E}[|X_n|] \propto \Delta t^{\frac{1}{2}}$. Intuitively, as trading becomes more frequent, there is less liquidity trading at each period to provide camouflage. Therefore, the informed traders trade less intensely and scale back the trading volume at each period. It can also be noted that in the monopolist case, the insider generates a negligible fraction of total trading volume, whereas in the imperfectly competitive case, the total aggregate volume submitted by the informed traders is comparable to the volume by the liquidity traders. The results are illustrated in Figure 4. When $M = 1$, the ratio quickly converges to zero when Δt is small. But the ratios converge to positive constants when $M > 1$ and can be arbitrarily close to $\frac{1}{2}$ when M is large enough. The empirical implications of this result is that, when the frequency of trading is sufficiently high,

Price Variance

Since the aggregate trading volume is comparable to the trading volume of the liquidity traders, following similar argument and Proposition 2.3, the informed traders' contribution to the total price variance can also be non-negligible. Figure 5 illustrates how the total price variance (blue lines) and its contribution by the informed traders (red lines) varies with the time interval between rounds of trading and the number of informed traders.

In the monopolist case, although total price variance increases when Δt is small, the contribution by the monopolistic trader converges to zero. The liquidity trader therefore contributes almost all of the price volatility near continuous trading. In the imperfectly competitive case, not only do informed traders contribute significantly to the total price variance near continuous trading, but the ratio increases as the number of informed traders increases.

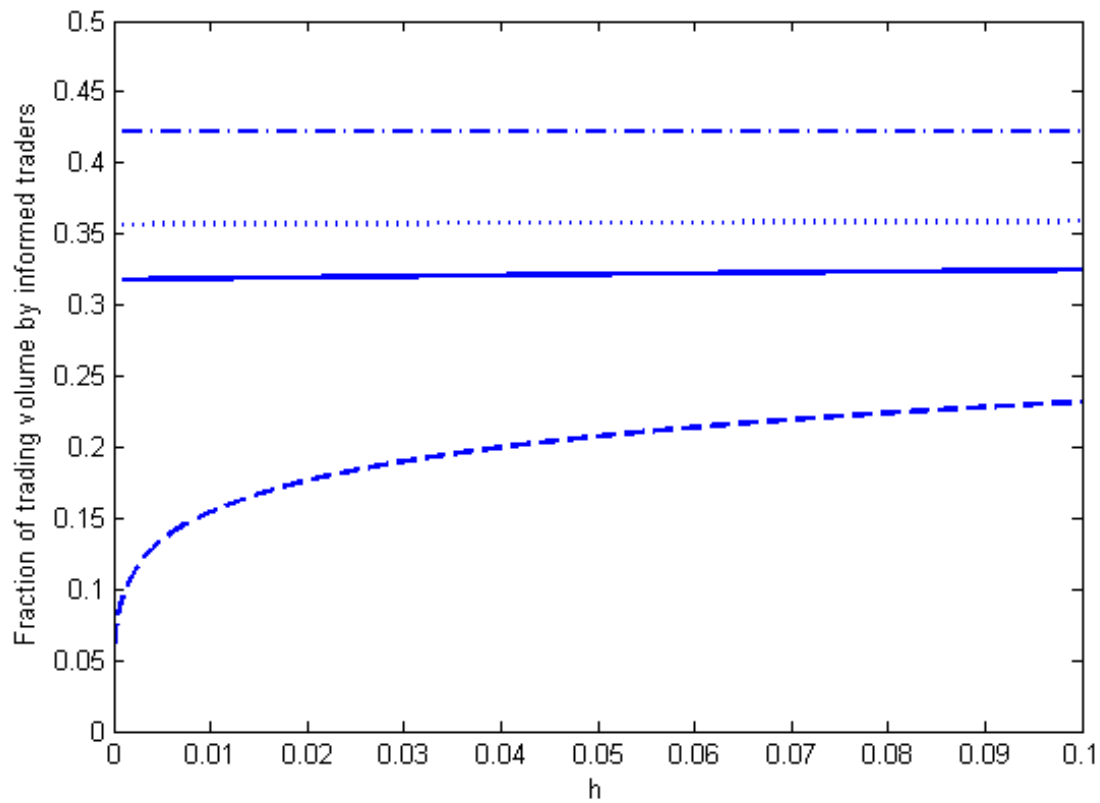


Figure 4: Fraction of trading volume of informed traders as a function of Δt for $M = 1, 2, 3$ and 10. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

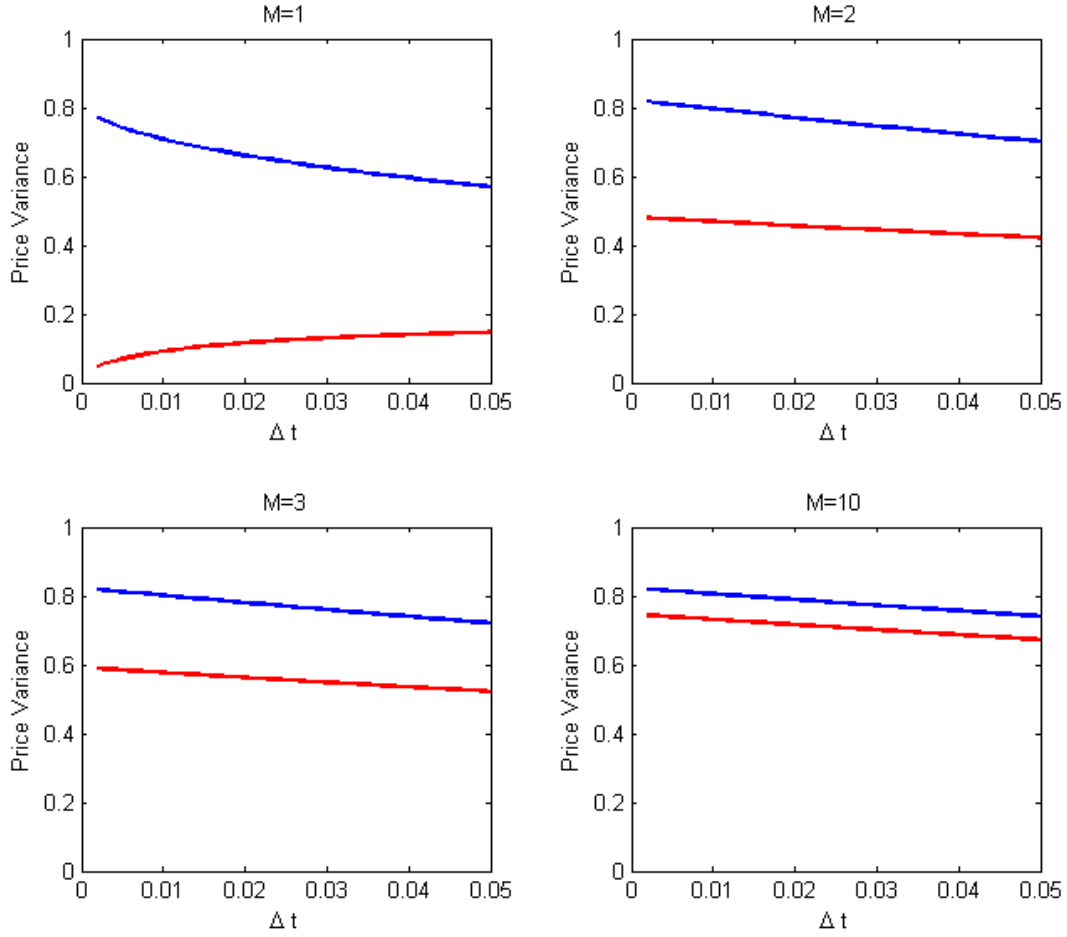


Figure 5: Total price variance (blue lines) and the contribution by the informed traders (red lines) as a function of Δt for $M = 1, 2, 3$ and 10 . Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

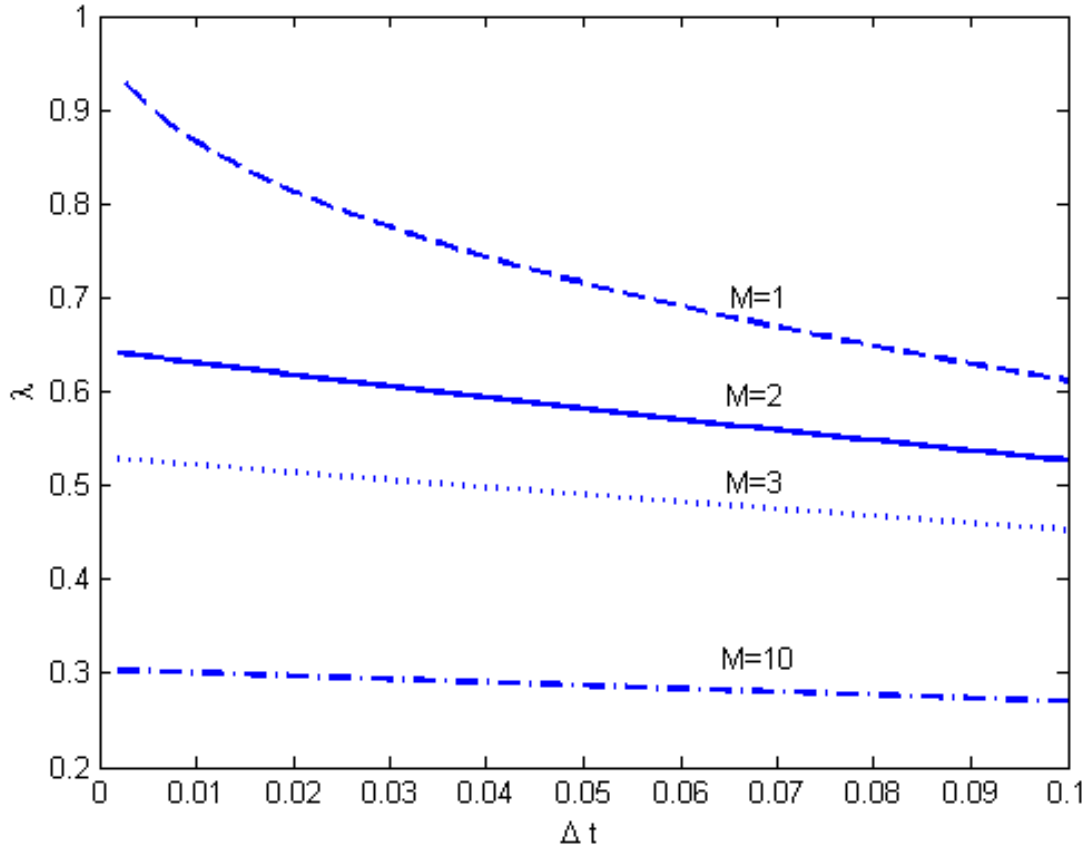


Figure 6: The market liquidity parameter λ as a function of Δt for $M = 1, 2, 3$ and 10 . Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

Market Liquidity and Profitability

Stationarity requires that the price impact λ is a time-independent constant. Figure 6 illustrates the effect of competition on λ . If we fix Δt and increase the number of informed traders, λ declines accordingly. This is because in a steady state, more competition leads to less information asymmetry between informed traders and market maker. When the number of informed traders is fixed and trading frequency increases, numerical calculations show that λ increases and converges to a positive constant.

The fact that λ remains strictly positive in the continuous trading limit ensures that

informed traders make strictly positive expected profits. This is because informed traders benefit from the trading of liquidity traders. Their profits are higher when price impact is higher and when liquidity traders are able to deviate the price further away from the efficient value giving informed traders opportunities to trade. To give a more rigorous explanation, remember that profit margin per share is $\sqrt{\Sigma_v}$ which is of order $\sqrt{\Delta t}$ when Δt is small. The demand submit in period n in absolute term by each informed trader is proportional to $\sqrt{\Delta t}$. Then at each period, each informed trader earns a small expected profits in the order of Δt . The present value of the aggregate profits at any period n is proportional to $\sum_{k=n}^{\infty} e^{-r(k-n)\Delta t} h$, which is finite when $\Delta t \rightarrow 0$. Hence, our model predicts that each informed trader can still earn positive expected profits in the continuous trading limit. But imperfect competition does make each informed trader worse off. To demonstrate the effect of competition on expected profits, I plot in Figure 7 the aggregate expected profits of informed traders of informed traders as a function of Δt for different M . When $M \geq 2$, the aggregate profits monotonically decrease with M for fixed Δt and converge to positive constants as $\Delta t \rightarrow 0$.

Figure 8 illustrates how the asymptotic price impact $\lim_{\Delta t \rightarrow 0} \lambda$ varies with the number of informed traders. $\lim_{\Delta t \rightarrow 0} \lambda$ monotonically decreases with M . When the number of traders is large enough, the market can be infinitely deep with $\lim_{\Delta t \rightarrow 0} \lambda$ very close to zero. Therefore, the aggregate profits of informed traders also converges to zero as $M \rightarrow \infty$.

Endogenous Information Acquisition and Policy Suggestion

I have just shown that when the number of informed traders (M) and the amount of noise trading (σ_u) are fixed, the market depth improves as the frequency of trading decreases. However, such result on market liquidity does not necessarily hold if we endogenize the

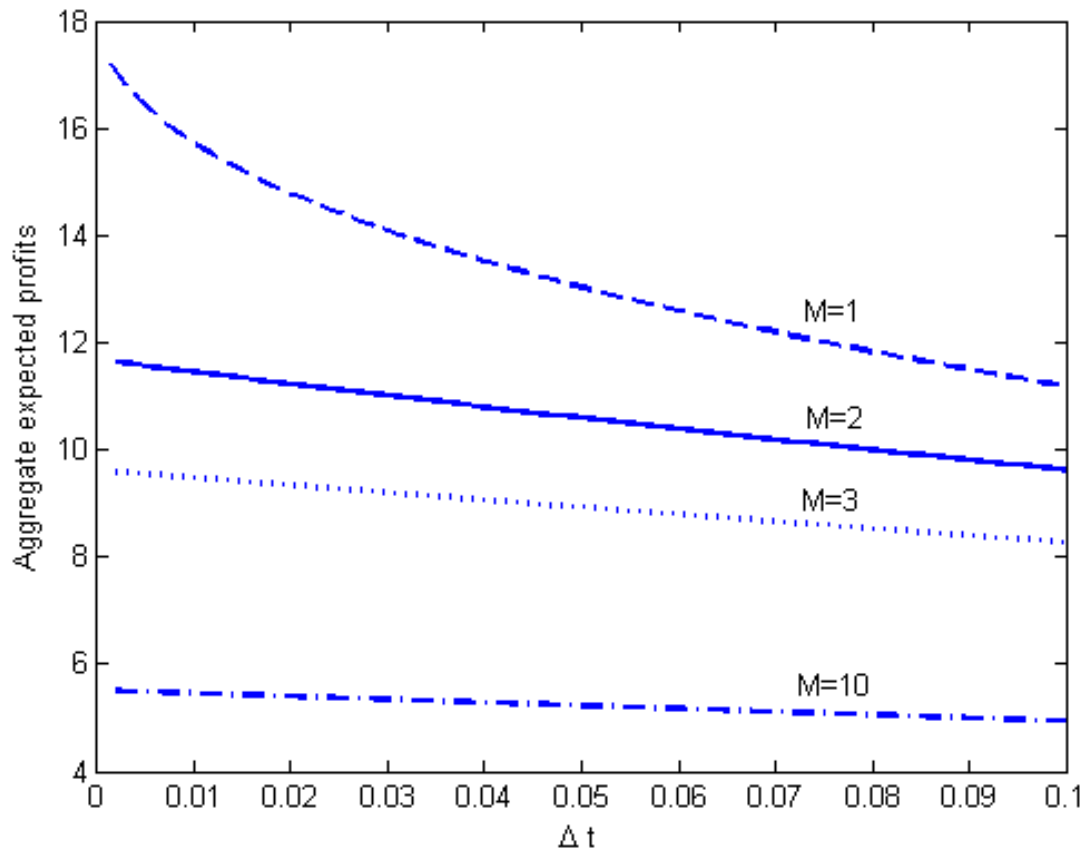


Figure 7: Aggregate expected profits as a function of Δt for $M = 1, 2, 3$ and 10 . Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

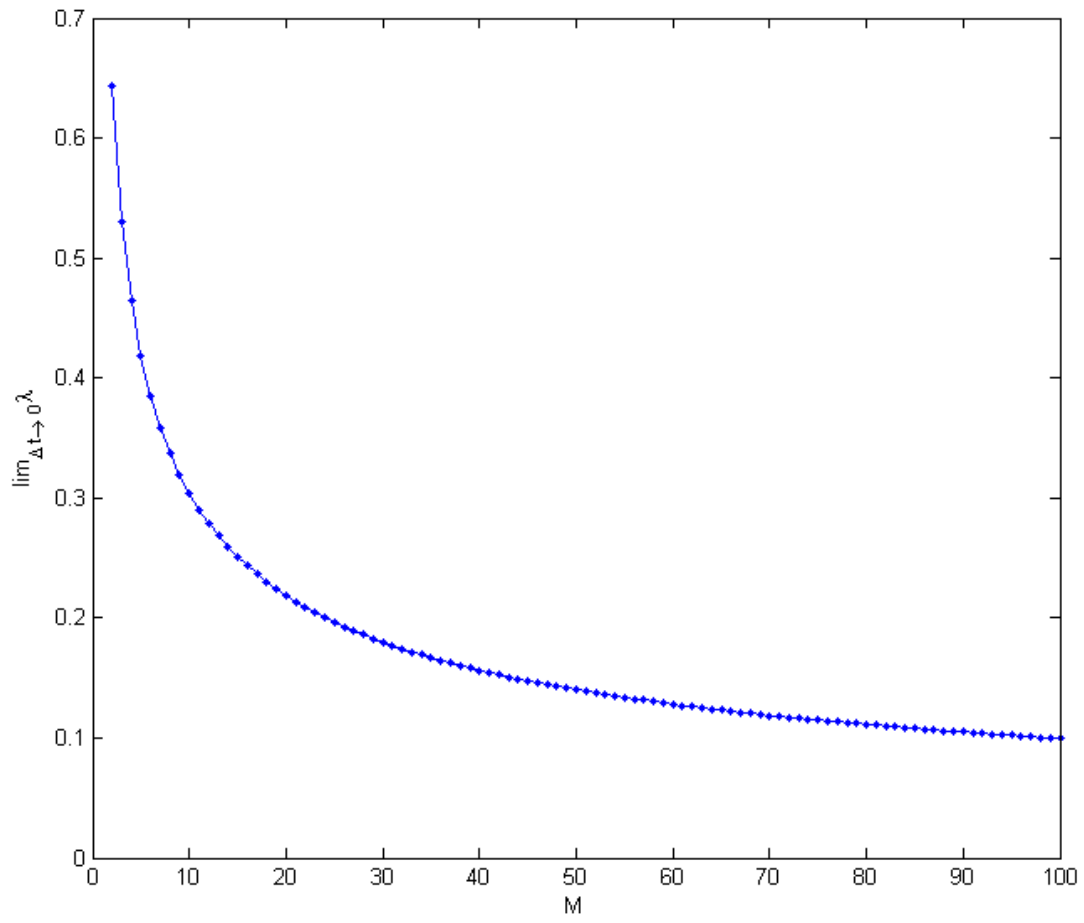


Figure 8: $\lim_{\Delta t \rightarrow 0} \lambda$ as a function of the number of informed traders.

number of informed traders by introducing a fixed cost to entry. The intuition is that as the trading frequency declines, the profitability of each informed trader drops as well. As the profitability reaches to zero, the number of informed traders in the market should decline to make sure the expected profits remain to be positive. Less competitive pressure results a less market depth provided by the market maker.

As shown in figure 9(b), with a positive fixed cost to entry c , initially the number of informed traders is $M = 4$. When the interval between trading increases to $\Delta t = 0.075$, the number of informed traders is reduced to three and there is a sudden jump in the price impact λ (figure 9(c)). Although the inverse measure of market efficiency Σ_v still monotonically increases with Δt (figure 9(d)), the market liquidity does not necessarily improve as Δt increases (worse by more than 10% at $\Delta t = 0.75$ in this example).

After the May 6th “flash crash”, HFTs have been under scrutiny and the subject of intense public debate and controversy. Regulators have expressed concerns over which whether HFTs affect the overall integrity of the equity markets. If HFTs were socially useless investments as argued by some economists, then it would be natural for regulators to impose a minimum latency (slowing the market). However, such latency requirement does not necessarily improve all the aspects of the markets. Take market liquidity for example, as shown in Figure 9, if orders are batched less frequently, the market depth might be worse off instead of being improved. This is because that traders’ profits will decrease if they have to trade less frequently, and as traders start to exit the market due to declining profits, the market becomes less liquid due to less competition among informed traders.

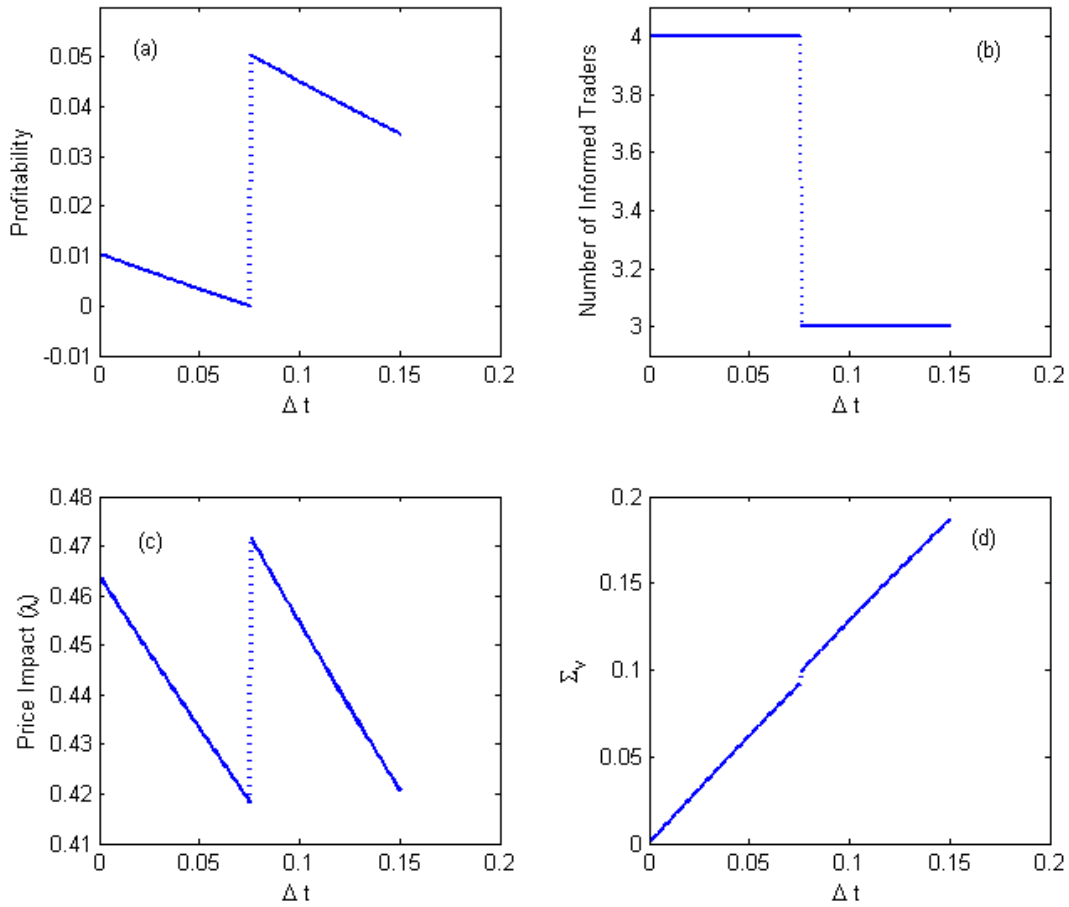


Figure 9: (a) Profitability, (b) number of informed traders (M), (c) price impact (λ) and (d) inverse measure of market efficiency (Σ_v) as functions of Δt when each informed trader has to pay a fixed cost c to enter the market. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$, $r = 0.05$ and $c = 0.1$.

5 Conclusion

In this paper, I analyze how imperfect competition among informed traders affects market efficiency, liquidity, trading volume and the profitability of informed traders. The combined trading of multiple informed traders is more aggressive than the monopolistic trader, the equilibrium price is even more revealing of the informed traders' private information, and market depth improves as the number of informed traders increases. In the continuous trading limit, the variance of private information held by informed traders goes to zero at a rate proportional to the time interval between rounds of trading. This is much faster than the corresponding strong form efficiency result in the Chau and Vayanos model, where the convergence rate is proportional to the square root of the time interval. In addition, in the limit as the time interval between rounds of trading goes to zero, the aggregate profits of the informed traders remain bounded away from zero and they contribute significantly to the total trading volume and price volatility.

If high frequency traders are “informed” in a sense that they are able to generate profitable private signals consistently by processing information from order flows and price movements of securities across market, then this model provides a reasonable characterization of those traders. My results suggest that the entry of more high frequency traders improves market efficiency by incorporating information more quickly into price, improves market liquidity by lowering price impact, and increases the fraction of trading volume from high frequency traders. But those traders remain profitable despite exploiting the same information set and implementing similar algorithms.

Future research can extend the results of this paper in two directions. First, to explain

why high frequency traders quickly reverse their inventories, we may add risk aversion. Second, the assumption that traders are identically informed is too strong. The assumption does not allow the more realistic scenario in which the informed traders learn from each other. Li (2012a) extends this paper by introducing a hierarchical information structure in which there is one strictly better informed trader and one less informed trader. Li (2012b) further relax the assumption in this paper even further to allow for a more general correlation among streams of private information in which each trader has to forecast the forecasts of other traders.

References

- Admati, Anat R and Paul Pfleiderer**, “A theory of intraday patterns: Volume and price variability,” *Review of Financial Studies*, 1988, 1 (1), 3–40.
- Back, Kerry**, “Insider Trading in Continuous Time,” *The Review of Financial Studies*, 1992, 5 (3), pp. 387–409.
- , **C. Henry Cao, and Gregory A. Willard**, “Imperfect Competition among Informed Traders,” *The Journal of Finance*, 2000, 55 (5), pp. 2117–2155.
- Breckenfelder, Johannes**, “Competition of High-Frequency Market Makers and Market Quality,” 2013.
- Chau, Minh and Dimitri Vayanos**, “Strong-Form Efficiency with Monopolistic Insiders,” *Review of Financial Studies*, September 2008, 21 (5), 2275–2306.
- Chlistalla, Michael**, “High frequency trading? Better than its reputation?,” *Deutsche Bank Research Briefing*, 2011.
- Foster, F. Douglas and S. Viswanathan**, “The Effect of Public Information and Competition on Trading Volume and Price Volatility,” *The Review of Financial Studies*, 1993, 6 (1), pp. 23–56.
- **and** – , “Strategic Trading with Asymmetrically Informed Traders and Long-Lived Information,” *Journal of Financial and Quantitative Analysis*, December 1994, 29 (04), 499–518.

– **and** –, “Strategic Trading When Agents Forecast the Forecasts of Others,” *The Journal of Finance*, 1996, 51 (4), pp. 1437–1478.

Hendershott, Terrence J. and Ryan Riordan, “High Frequency Trading and Price Discovery,” *SSRN eLibrary*, 2011.

Holden, Craig W and Avanidhar Subrahmanyam, “Long-Lived Private Information and Imperfect Competition,” *Journal of Finance*, March 1992, 47 (1), 247–70.

Kirilenko, Andrei A., Albert (Pete) S. Kyle, Mehrdad Samadi, and Tugkan Tuzun, “The Flash Crash: The Impact of High Frequency Trading on an Electronic Market,” *SSRN eLibrary*, 2011.

Kyle, Albert S, “Continuous Auctions and Insider Trading,” *Econometrica*, November 1985, 53 (6), 1315–35.

Li, Su, “Speculative dynamics II: Asymmetric Informed Traders,” 2012.

–, “Speculative dynamics III: When Traders Forecast the Forecasts of Others,” 2012.

Martinez, Victor H. and Ioanid Rosu, “High Frequency Traders, News and Volatility,” 2011.

Rosu, Ioanid, “A Dynamic Model of the Limit Order Book,” *Review of Financial Studies*, 2009, 22 (11), 4601–4641.

Appendix A

Proof of Lemma 2.1: It is easy to verify that

$$v_{n'} = (1 - \kappa\Delta t)^{n'-n}v_n + \kappa\Delta t\bar{v} \sum_{\tau=n}^{n'} (1 - \kappa\Delta t)^{n'-\tau} + \sum_{\tau=n}^{n'} (1 - \kappa\Delta t)^{n'-\tau}\epsilon_{v,\tau}. \quad (48)$$

Taking expectations in equation (48), we have

$$\mathbb{E}[v_{n'}|I_n^m] = (1 - \kappa\Delta t)^{n'-n} \mathbb{E}[v_n|I_n^m] + \kappa\Delta t\bar{v}(1 - (1 - \kappa\Delta t)^{n'-n+1}). \quad (49)$$

Substituting into equation (2), the price is equal to

$$\begin{aligned} p_n &= \sum_{n'=n}^{+\infty} 1 - \exp(-r\Delta t) \exp(-r(n' - n)\Delta t) ((1 - \kappa\Delta t)^{n'-n} \mathbb{E}[v_n|I_n^m] \\ &\quad + \kappa\Delta t\bar{v}(1 - (1 - \kappa\Delta t)^{n'-n+1})) \\ &= \frac{(1 - \exp(-r\Delta t))}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} \mathbb{E}[v_n|I_n^m] + \frac{\kappa\Delta t\bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)}. \end{aligned} \quad (50)$$

Proof of Lemma 2.2: First, one can find that:

$$\begin{aligned}
& \sum_{n'=n}^{+\infty} \exp(-r(n'-n)\Delta t) \left(\sum_{\tau=n}^{n'} x_{j,\tau} (v_{n'} - p_\tau) \right) = \tag{51} \\
& x_{j,n} \sum_{\tau=n}^{+\infty} \exp(-r(\tau-n)\Delta t) (v_\tau - p_n) \\
& + x_{j,n+1} \sum_{\tau=n+1}^{+\infty} \exp(-r(\tau-n)\Delta t) (v_\tau - p_{n+1}) \\
& + x_{j,n+2} \sum_{\tau=n+1}^{+\infty} \exp(-r(\tau-n)\Delta t) (v_\tau - p_{n+2}) + \dots \\
& = x_{j,n} * \left(\sum_{\tau=n}^{+\infty} \exp(-r(\tau-n)\Delta t) v_\tau - \frac{p_n}{1 - \exp(-r\Delta t)} \right) \\
& + x_{j,n+1} * \left(\sum_{\tau=n+1}^{+\infty} \exp(-r(\tau-n)\Delta t) v_\tau - \frac{\exp(-r\Delta t)p_n}{1 - \exp(-r\Delta t)} \right) \\
& + x_{j,n+2} * \left(\sum_{\tau=n+2}^{+\infty} \exp(-r(\tau-n)\Delta t) v_\tau - \frac{\exp(-2r\Delta t)p_n}{1 - \exp(-r\Delta t)} \right) + \dots \tag{52}
\end{aligned}$$

Substituting equation (52) into equation (6) and using the result in equation (48), we

have

$$\begin{aligned}
& E \left[\sum_{\tau=n}^{+\infty} (1 - \exp(-r\Delta t)) \exp(-r(n'-n)\Delta t) \left(\sum_{\tau=n}^{n'} x_{j,\tau} (v_{n'} - p_\tau) \right) \middle| I_{n-1}^i \right] = \tag{53} \\
& E \left[x_{j,n} \left(\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} v_n + \frac{\kappa\Delta t \bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} - p_n \right) \right. \\
& + \exp(-r\Delta t) x_{j,n+1} \left(\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} v_{n+1} + \frac{\kappa\Delta t \bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} - p_{n+1} \right) \\
& + \exp(-2r\Delta t) x_{j,n+1} \left(\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} v_{n+1} + \frac{\kappa\Delta t \bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} - p_{n+2} \right) \\
& \left. \dots \middle| I_{n-1}^i \right] \\
& = E \left[\sum_{n'=n}^{+\infty} \exp(-r(n'-n)\Delta t) \left(\frac{1 - \exp(-r\Delta t)}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} v_{n'} + \frac{\kappa\Delta t \bar{v}}{1 - \exp(-r\Delta t)(1 - \kappa\Delta t)} - p_{n'} \right) \middle| I_{n-1}^i \right].
\end{aligned}$$

Appendix B

Proof of Lemma 2.3: We first compute each component of the covariance matrix of the vector (v_{n-1}, y_n) conditional on the market maker's information set I_{n-1}^m :

$$\text{Cov}(y_n, v_{n-1} | I_{n-1}) = \text{Cov}\left(\sum_{j=1}^M \beta_j (v_{n-1} - \hat{v}_{n-1}), v_{n-1} | I_{n-1}\right) = \sum_{j=1}^M \beta_j \text{Var}(v_{n-1} | I_{n-1}) = \sum_{j=1}^M \beta_j \Sigma_v \quad (54)$$

$$\text{Var}(y_n | I_{n-1}^m) = \left(\sum_{j=1}^M \beta_j\right)^2 \Sigma_v + \sigma_u^2 \Delta t. \quad (55)$$

Then under the market maker's belief, we have the joint distribution of $(v_{n-1}, y_n)'$

$$\begin{pmatrix} v_{n-1} \\ y_n \end{pmatrix} \sim N\left(\begin{pmatrix} \hat{v}_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_v & \sum_{j=1}^M \beta_j \Sigma_v \\ \sum_{j=1}^M \beta_j \Sigma_v & (\sum_{j=1}^M \beta_j)^2 \Sigma_v + \sigma_u^2 \Delta t \end{pmatrix} \right). \quad (56)$$

Then applying the projection theorem, we have

$$\frac{\lambda}{1 - \kappa \Delta t} = \left((\sum_{j=1}^M \beta_j)^2 \Sigma_v + \sigma_u^2 \Delta t \right)^{-1} \sum_{j=1}^M \beta_j \Sigma_v \quad (57)$$

which can be reduced to equation (14).

Applying the projection theorem again, we can derive the variance of η_n

$$\begin{aligned} \text{Var}(\eta_n) &= \text{Var}(v_{n-1} | I_{n-1}) - \frac{\lambda}{1 - \kappa \Delta t} \sum_{j=1}^M \beta_j \Sigma_v \\ &= \Sigma_v - \frac{\lambda}{1 - \kappa \Delta t} \sum_{j=1}^M \beta_j \Sigma_v \\ &= \frac{\Sigma_v \sigma_u^2 \Delta t}{\Sigma_v (\sum_{j=1}^M \beta_j)^2 + \sigma_u^2 \Delta t}. \end{aligned} \quad (58)$$

The uncertainty of market maker's posterior belief about v_n is given by

$$\text{Var}(v_n|I_n^m) = \text{Var}((1 - \kappa\Delta t)\eta_n + \epsilon_{v,n}) = (1 - \kappa\Delta t)^2\text{Var}(\eta_n) + \sigma_v^2\Delta t. \quad (59)$$

By stationary condition, we must have $\text{Var}(v_n|I_n^m) = \Sigma_v$ which leads to equation (15).

Proof of Lemma 2.4: From equations (1) and (12), market maker's estimation error on v_n is

$$v_n - \hat{v}_n = (1 - \kappa\Delta t)(v_{n-1} - \hat{v}_{n-1}) - \lambda y_n + \epsilon_{v,n}. \quad (60)$$

Substituting for $v_n - \hat{v}_n$ in equation (18), we find

$$\begin{aligned} V_j(v_{n-1}, \hat{v}_{n-1}) &= \max_{x_{j,n}}(x_{i,j,n}((1 - \kappa\Delta t)(v_{n-1} - \hat{v}_{n-1}) - \lambda(x_{j,n} + \sum_{i \neq j} x_{i,n}))) \\ &+ e^{-r\Delta t}(B((1 - \kappa\Delta t)(v_{n-1} - \hat{v}_{n-1}) - \lambda(x_{j,n} + \sum_{i \neq j} x_{i,n}))^2 \\ &+ \lambda^2\sigma_u^2\Delta t + \sigma_v^2\Delta t + C)). \end{aligned} \quad (61)$$

The first order condition yields

$$x_{j,n} = \frac{(1 - 2e^{-r\Delta t}B\lambda)(1 - \kappa\Delta t)(v_{n-1} - \hat{v}_{n-1}) + \lambda(2e^{-r\Delta t}B\lambda - 1)\sum_{i \neq j} x_{i,n}}{2\lambda(1 - e^{-r\Delta t}B\lambda)}. \quad (62)$$

The second order condition requires that

$$e^{-r\Delta t}B\lambda - 1 < 0. \quad (63)$$

Because of symmetry argument, the only possible equilibrium is one in which their strategies are identical. We should have $x_{i,n} = x_{j,n}$ for $i \neq j$, which leads

$$x_{j,n} = \frac{(1 - 2e^{-r\Delta t}B\lambda)(1 - \kappa\Delta t)}{\lambda(M + 1 - 2Me^{-r\Delta t}B\lambda)}(v_{n-1} - \hat{v}_{n-1}) = \beta(v_{n-1} - \hat{v}_{n-1}) \quad (64)$$

where

$$\beta = \frac{(1 - 2e^{-r\Delta t}B\lambda)(1 - \kappa\Delta t)}{\lambda(M + 1 - 2Me^{-r\Delta t}B\lambda)}. \quad (65)$$

Substituting for $x_{j,n}$ back in the Bellman equation and matching the $(v_{n-1} - \hat{v}_{n-1})^2$ term and constant term, we find

$$B = \beta(1 - \kappa\Delta t - \lambda M\beta) + e^{-r\Delta t}B(1 - \kappa\Delta t - \lambda M\beta)^2 \quad (66)$$

which can be reduced to

$$B = \frac{(1 - \kappa h)^2(1 - e^{-r\Delta t}B\lambda)}{\lambda(1 + M - 2Me^{-r\Delta t}B\lambda)^2} \quad (67)$$

and

$$C = \frac{e^{-r\Delta t}B(\lambda^2\sigma_u^2 + \sigma_v^2)\Delta t}{1 - e^{-r\Delta t}}. \quad (68)$$

Proof of Proposition 2.1:

First, we define $q = e^{-r\Delta t}\lambda B$ and $Z = e^{-r\Delta t}(1 - \kappa\Delta t)^2$. From equation (20), we have

$$f(q) = 4M^2q^3 - 4M(M + 1)q^2 + ((M + 1)^2 + Z)q - Z = 0. \quad (69)$$

The cubic equation $f(q)$ has three real roots:

$$\begin{aligned} q_1 &= \frac{M + 1}{3M} \\ &- \frac{1}{6M} \sqrt[3]{(M + 1)^3 + (9 - 18M)Z + \sqrt{((M + 1)^3 + (9 - 18M)Z)^2 - ((M + 1)^2 - 3Z)^3}} \\ &- \frac{1}{6M} \sqrt[3]{(M + 1)^3 + (9 - 18M)Z + \sqrt{((M + 1)^3 + (9 - 18M)Z)^2 + ((M + 1)^2 - 3Z)^3}} \end{aligned} \quad (70)$$

$$\begin{aligned}
q_2 &= \frac{M+1}{3M} & (71) \\
&- \frac{1+i\sqrt{3}}{12M} \sqrt[3]{(M+1)^3 + (9-18M)Z + \sqrt{((M+1)^3 + (9-18M)Z)^2 - ((M+1)^2 - 3Z)^3}} \\
&- \frac{1-i\sqrt{3}}{12M} \sqrt[3]{(M+1)^3 + (9-18M)Z + \sqrt{((M+1)^3 + (9-18M)Z)^2 + ((M+1)^2 - 3Z)^3}}
\end{aligned}$$

and

$$\begin{aligned}
q_3 &= \frac{M+1}{3M} & (72) \\
&- \frac{1-i\sqrt{3}}{12M} \sqrt[3]{(M+1)^3 + (9-18M)Z + \sqrt{((M+1)^3 + (9-18M)Z)^2 - ((M+1)^2 - 3Z)^3}} \\
&- \frac{1+i\sqrt{3}}{12M} \sqrt[3]{(M+1)^3 + (9-18M)Z + \sqrt{((M+1)^3 + (9-18M)Z)^2 + ((M+1)^2 - 3Z)^3}}.
\end{aligned}$$

It can be easily verified that for $M > 1$, we have

$$f(0) = -Z < 0, f(1/2) = > 0, f(1) = (M-1)^2 > 0. \quad (73)$$

In addition, the quadratic function $f'(q) = 12M^2q^2 - 8M(M+1)q + (M+1)^2 + Z = 0$ has two solutions,

$$q_{4,5} = \frac{M+1}{3M} \pm \frac{\sqrt{(M+1)^2 - 3Z}}{6M}. \quad (74)$$

It can be easily verified that q_4 and q_5 satisfy the following conditions:

$$\begin{aligned}
q_4 &\in (0, \frac{1}{2}) & (75) \\
q_5 &\in (\frac{1}{2}, 1)
\end{aligned}$$

Equations (73) and (76) imply that $0 < q_1 < \frac{1}{2}$ and $\frac{1}{2} < q_2 < q_3 < 1$.

From the second order condition (equation (63)) we have $q < 1$. In addition, from equation (19) any root that makes economically feasible must lie in the range $q \in (0, \frac{1}{2})$.

The only possible solution is therefore q_1 . From equation (14) and equation (14), equation

(19) can be rewritten as

$$\frac{\lambda\beta}{1 - \kappa\Delta t} = \frac{\Sigma_v M \beta^2}{\Sigma_v M^2 \beta^2 + \sigma_u^2 \Delta t} = \frac{1 - 2q_1}{1 + M(1 - 2q_1)}. \quad (76)$$

From equation (76), one can find that

$$\beta = \sqrt{\frac{\sigma_u^2 (1 - 2q_1) \Delta t}{\Sigma_v M}}. \quad (77)$$

Substituting equation (77) into equation (15), we can find the expression for Σ_v

$$\Sigma_v = \frac{(M(1 - 2q_1) + 1)\sigma_v^2 \Delta t}{M(1 - 2q_1) + 1 - (1 - \kappa\Delta t)^2}. \quad (78)$$

Then from equations (14, 21, 77) and expression for q_1 , we can derive the remaining parameters β , λ , B and C , respectively with the expressions for β and λ given by:

$$\beta = \frac{\sigma_u}{\sigma_v} \sqrt{\frac{(1 - 2q_1)(M(1 - 2q_1) + 1 - (1 - \kappa\Delta t)^2)}{M(M(1 - 2q_1) + 1)}} \quad (79)$$

$$\lambda = \frac{(1 - \kappa\Delta t)\sigma_v \sqrt{M(1 - 2q_1)}}{\sigma_u \sqrt{M(1 - 2q_1) + 1} \sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}}, \quad (80)$$

$$B = \frac{e^{r\Delta t} q_1 \sigma_u \sqrt{M(1 - 2q_1) + 1} \sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}}{(1 - \kappa\Delta t)\sigma_v \sqrt{M(1 - 2q_1)}} \quad (81)$$

and

$$C = \frac{q_1 \sigma_u \sigma_v \Delta t}{1 - e^{-r\Delta t}} \left(\frac{(1 - \kappa\Delta t) \sqrt{M(1 - 2q_1)}}{\sqrt{M(1 - 2q_1) + 1} \sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}} \right. \\ \left. + \frac{\sqrt{M(1 - 2q_1) + 1} \sqrt{1 + M(1 - 2q_1) - (1 - \kappa\Delta t)^2}}{(1 - \kappa\Delta t) \sqrt{M(1 - 2q_1)}} \right) \quad (82)$$

Proof of Proposition 2.2: We first define

$$S_v = \Sigma_v / \Delta t, \quad (83)$$

and

$$\begin{aligned}
q_0 &= \frac{M+1}{3M} \\
&- \frac{1}{6M} \sqrt[3]{(M+1)^3 - 18M - 9 + \sqrt{((M+1)^3 - 18M - 9)^2 - (M^2 + 2M - 2)^3}} \\
&- \frac{1}{6M} \sqrt[3]{(M+1)^3 - 18M - 9 - \sqrt{((M+1)^3 - 18M - 9)^2 - (M^2 + 2M - 2)^3}}.
\end{aligned} \tag{84}$$

Then, as Δt approaches 0, equations (76) and (15) become:

$$\frac{M^2 S_v \beta^2}{M^2 S_v \beta^2 + \sigma_u^2} = \frac{M(1 - 2q_0)}{1 + M(1 - 2q_0)} \tag{85}$$

$$\frac{S_v \sigma_u^2}{M^2 \beta^2 S_v + \sigma_u^2} + \Sigma_v^2 = S_v. \tag{86}$$

The set of the above nonlinear equations has the solution

$$S_v = \frac{\sigma_v^2(1 + M(1 - 2q_0))}{M(1 - 2q_0)} \tag{87}$$

and

$$\beta = \frac{\sigma_u(1 - 2q_0)}{\sigma_v \sqrt{1 + M(1 - 2q_0)}}. \tag{88}$$

By the continuity argument, the limiting results of Σ_v and β become

$$\lim_{\Delta t \rightarrow 0} \frac{\Sigma_v}{\Delta t} = S_v = \frac{\sigma_v^2(1 + M(1 - 2q_0))}{M(1 - 2q_0)} \tag{89}$$

and

$$\lim_{\Delta t \rightarrow 0} \beta = \frac{\sigma_u(1 - 2q_0)}{\sigma_v \sqrt{1 + M(1 - 2q_0)}}. \tag{90}$$

Then, from equations (14), (76) and (21), we derive the asymptotic results for λ , B and C :

$$\lim_{\Delta t \rightarrow 0} \lambda = \frac{\sigma_v}{\sigma_u} \sqrt{\frac{1}{1 + M(1 - 2q_0)}} \tag{91}$$

$$\lim_{\Delta t \rightarrow 0} B = \frac{q_0 \sigma_u}{\sigma_v} \sqrt{1 + M(1 - 2q_0)} \quad (92)$$

$$\lim_{\Delta t \rightarrow 0} C = q_0 \sigma_u \sigma_v (\sqrt{1 + M(1 - 2q_0)} + \sqrt{\frac{1}{1 + M(1 - 2q_0)}}). \quad (93)$$

Proof of Proposition 2.3:

Over one trading period, the absolute aggregate demand of the informed traders at the n^{th} trading period is Mx_n while the demand of the liquidity traders is u_n . Following Admati and Pfleiderer (1988), define the total trading volume $Vol_{total} = M|x_n| + |u_n| + |y_n|$. The fraction of contribution by the informed traders ξ_M is therefore

$$\xi_M = \frac{M|x_n|}{M|x_n| + |u_n| + |y_n|}. \quad (94)$$

Since $|x_n| = \sqrt{\frac{2}{\pi} \text{Var}(x_n)}$, $|y_n| = \sqrt{\frac{2}{\pi} \text{Var}(y_n)}$ and $|u_n| = \sqrt{\frac{2}{\pi} \text{Var}(u_n)}$, ξ_M can be written as

$$\xi_M = \frac{M\sqrt{\text{Var}(x_n)}}{M\sqrt{\text{Var}(x_n)} + \sqrt{\text{Var}(y_n)} + \sqrt{\text{Var}(u_n)}}, \quad (95)$$

where

$$\begin{aligned} \text{Var}(x_n) &= \beta^2 \text{Var}(v_{n-1} - \hat{v}_{n-1}) = \beta^2 \Sigma_v \\ &= (1 - 2q_1) \sigma_u^2 \Delta t, \end{aligned} \quad (96)$$

$$\text{Var}(u_n) = \sigma_u^2 \Delta t, \quad (97)$$

and

$$\text{Var}(y_n) = M^2 \text{Var}(x_n) + \text{Var}(u_n) = (M^2(1 - 2q_1) + 1) \sigma_u^2 \Delta t. \quad (98)$$

Substituting $\text{Var}(x_n)$, $\text{Var}(u_n)$ and $\text{Var}(y_n)$ into equation 95, we have

$$\xi_M = \frac{M\sqrt{1 - 2q_1}}{M\sqrt{1 - 2q_1} + 1 + \sqrt{M^2(1 - 2q_1) + 1}} \quad (99)$$

which depends only on the number of informed traders M and the time interval between trading (q_1 is a function of Δt).

In the limit of continuous trading ($\Delta t \rightarrow 0$), we have $\lim_{\Delta t \rightarrow 0} q_1 = q_0$ and therefore

$$\lim_{\Delta t \rightarrow 0} \xi_M = \frac{M\sqrt{1-2q_0}}{M\sqrt{1-2q_0} + 1 + \sqrt{M^2(1-2q_0) + 1}}. \quad (100)$$